# Statistics 655 Comprehensive Written Exam
# August 2019

All problems have equal weight; partial credit will be given for each part of a problem. In budgeting your time, note that some parts will take longer than others. In many cases, different parts of a problem can be done independently, so if you are unsure how to handle one part of a problem, don't hesitate to try the others. No answer should require a great deal of computation or a complicated proof.

1. Let $X_1, X_2, \ldots$ be i.i.d. positive random variables having probability density function $f$ such that $f(x_n) \to \lambda$ for any decreasing sequence $x_1 \geq x_2 \geq \cdots > 0$ tending to 0.

(a) Let $F$ be the CDF of $X_1$. What can you say about the value of $F(x)$ when $x$ is positive and close to zero?

(b) Show that the sequence $Y_n = n \min\{X_1, \ldots, X_n\}$, $n \geq 1$, has a weak limit, and identify the limiting distribution.

2. Let $X_1, X_2, \ldots \in \mathbb{R}^d$ be i.i.d. random vectors with mean $\mathbb{E}X_i = \mu$ and variance matrix $\mathrm{Var}(X_i) > 0$. Let

$$T_n^2 = (n-1)(\overline{X}_n - \mu)^t S_n^{-1} (\overline{X}_n - \mu)$$

be Hotelling's $T^2$ statistic, where $S_n = n^{-1} \sum_{i=1}^n (X_i - \overline{X}_n)(X_i - \overline{X}_n)^t$ is the sample variance matrix. Find the limiting distribution of $T_n^2$. Justify your answer.

3. Let $V \subseteq \mathbb{R}^n$ be a finite set of vectors $v = (v_1, \ldots, v_n)^t$ with $L = \max_{v \in V} ||v||_2$, and let $\varepsilon_1, \ldots, \varepsilon_n$ be independent sign variables with $P(\varepsilon_i = 1) = P(\varepsilon_i = -1) = 1/2$.

(a) Find a bound on the moment generating functions of the random variables $\sum_{i=1}^n \varepsilon_i v_i$ in terms of the constant $L$.

(b) Establish the following inequality, where $|V|$ denotes the cardinality of $V$. You may appeal to results from class, but state them as carefully as you can.

$$\mathbb{E}\left[\max_{v \in V} \sum_{i=1}^n \varepsilon_i v_i\right] \;\leq\; \sqrt{2L^2 \log |V|}$$

4. Let $U_1, U_2, \ldots, U \in \mathbb{R}$ and $V_1, V_2, \ldots V \in \mathbb{R}$ be random variables and let $X_1, X_2, \ldots, X \in \mathbb{R}^d$ be random vectors, all defined on the same probability space. Suppose that $U_n \to U$ in probability, $V_n \to V$ in probability where $P(V > 0) = 1$, and $X_n \Rightarrow X$ in law. In each of the following cases, (i) indicate whether the quantity converges as $n$ tends to infinity, (ii) identify the type of convergence, and (iii) identify the limit if one exists. Carefully justify your answers. You may appeal to general results, but should establish counterexamples where appropriate.

(a) $(U_n - U)^2 X_n$

(b) $U_n^2 \log V_n$

(c) $V_n X_n$

(d) $\exp\{-||X_n||\}$

(e) $\mathbb{E} \exp\{-||X_n||\}$

(f) $(V_n X_n)/V$