STOR 635 **Exam: CWE** **Year:** 2019

There are 4 questions. Attempt all questions. You may appeal to any result proved in class without proof unless you are specifically asked to "**Give a complete proof.** ". State any result you use. All questions are worth the same number of total points (10 points). Points for parts of a question can be found in boxes on the right. *Even if you don't know the complete solution DONT STRESS OUT. Put down your attempt.* Partial credit will be assigned so try your best on each question.

1. Fix a probability mass function $\mathbf{p} := \{p_k\}_{k\geqslant 0}$ and let $\{Z_n\}_{n\geqslant 0}$ with $Z_0 = 1$ be a branching process with offspring distribution $\mathbf{p}$. More precisely, let $\{\xi_{i,j} : i, j \geqslant 1\}$ be **i.i.d** with distribution $\mathbf{p}$. Now define the sequence $\{Z_n\}_{n\geqslant 0}$ recursively with $Z_0 = 1$ and let

$$Z_n := \sum_{j=1}^{Z_{n-1}} \xi_{n,j} \qquad n \geqslant 1,$$

with the understanding that if $Z_{n-1} = 0$ then $Z_n = 0$. Thus $\xi_{n,j}$ is interpreted as the number of children of individual j in generation $n - 1$. Define the probability generating function of $\mathbf{p}$ as

$$\phi(s) := \sum_{k=0}^{\infty} s^k p_k, \qquad s \in [0, 1]$$

Suppose there exists a unique $0 < \rho < 1$ such that $\phi(\rho) = \rho$.

(a) Show that the sequence $\left\{\rho^{Z_n} : n \geqslant 0\right\}$ is a Martingale. Here the filtration $\{\mathcal{F}_n : n \geqslant 0\}$ is defined by $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and for $n \geqslant 1$, $\mathcal{F}_n = \sigma(\{\xi_{mj} : 1 \leqslant m \leqslant n, j \geqslant 1\})$. **Give a complete proof.**

[5]

(b) Give reasons why there should be a limit random variable Y such that

$$\rho^{Z_n} \xrightarrow{a.s.} Y.$$

[3]

(c) Calculate $\mathbb{E}(Y)$. Give reasons for your answer (don't just put down the answer). [2]

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2. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\{\mathcal{F}_n : n \geqslant 1\}$ be a collection of increasing sub σ-fields namely $\mathcal{F}_n \subseteq \mathcal{F}$ for all $n \geqslant 1$ and

$$\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \mathcal{F}_3 \subseteq \cdots$$

Think of $\mathcal{F}_n$ as the amount of information on day n. Define the sigma field

$$\mathcal{F}_\infty := \sigma(\cup_{n=1}^{\infty} \mathcal{F}_n).$$

Give a complete proof to show that for any $A \in \mathcal{F}_\infty$, and any $\varepsilon > 0$, you can find an $n < \infty$ and $B \in \mathcal{F}_n$ (both n and B typically depend on A and ε) such that

$$\mathbb{P}(A\Delta B) \leqslant \varepsilon.$$

Here $B\Delta A := (B \setminus A) \cup (A \setminus B)$.

Hint: Good set principle

Point of the problem: You are showing that you can **approximate** any set in $\mathcal{F}_\infty$ by sets that you know about in "finite" time.

[10]

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3. Suppose the Markov chain has finite state space S with $|S| = K$. Further assume it is irreducible namely for every $x, y \in S$, $\mathbb{P}_x(T_y < \infty) = 1$ where $T_y = \inf\{n \geqslant 1 : X_n = y\}$. Let $a := \inf\{p_{xy} : p_{xy} > 0\}$ i.e. the smallest value amongst all strictly positive elements of the transition matrix. Show that there exists a constant $C(a, K) < \infty$ such that
$$\max_{x,y \in S} \mathbb{E}_x(T_y) \leqslant C(a, K).$$

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4. Suppose $\{X_i : i \geqslant 1\}$ are a sequence of real valued **exchangeable** (not necessarily integrable i.e. one could have $\mathbb{E}(|X_1|) = \infty$) random variables. Further suppose $\varphi : \mathbb{R}_+ \to \mathbb{R}$ is a bounded measurable function. Give a **complete proof** to show that there exists a random variable Y_∞ such that
$$\frac{\sum_{i=1}^n \varphi(X_i)}{n} \xrightarrow{a.s.} Y_\infty$$
If you are planning to use De-Finetti's theorem then give a complete proof to show how one can derive the result above from De-Finetti's theorem. Just writing down De-Finetti implies "conditionally *i.i.d.* and thus we get the above result " will get you at most 3 points unless you can clearly justify how you can use De-Finetti to prove the above result. Note: you can also prove the above result directly without appealing to De-Finetti.

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