STOR 634, CWE 2018-19.

Each problem is 10 points. There are 5 problems in all.

1. (*10 points*) Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\{A_n\}$ be a sequence of events in $\mathcal{F}$.

a. (*3 pts*) Give the definition of $\limsup A_n$ and $\liminf A_n$. Show that $\liminf A_n \subset \limsup A_n$.

b. (*3 pts*) Show that $1_{\limsup A_n} = \limsup 1_{A_n}$.

c. (*4 pts*) Suppose that $\{A_n\}$ are independent. Show that $P(\limsup(A_n \cup A_{n+1})) \in \{0, 1\}$.

2. (*10 points*) Let $X_{n,k}$, $1 \le k \le n$, $n \ge 1$ be real random variables on some probability space $(\Omega, \mathcal{F}, P)$. For each n, let $X_{n,k}$, $1 \le k \le n$ be independent. Let $b_n > 0$ with $b_n \to \infty$. Let $\bar{X}_{n,k} \doteq X_{n,k} 1_{\{|X_{n,k}| \le b_n\}}$. Suppose that as $n \to \infty$ (i) $\sum_{k=1}^n P(|X_{n,k}| > b_n) \to 0$, and (ii) $b_n^{-2} \sum_{k=1}^n E\bar{X}_{n,k}^2 \to 0$. Let $S_n = X_{n,1} + \cdots + X_{n,n}$ and $a_n = \sum_{k=1}^n E\bar{X}_{n,k}$.

a. (*3 pts*) Let $\bar{S}_n \doteq \bar{X}_{n,1} + \cdots + \bar{X}_{n,n}$. Show that $P(S_n \ne \bar{S}_n)$ converges to 0 as $n \to \infty$.

b. (*3 pts*) Show that $(\bar{S}_n - a_n)/b_n$ converges to 0 in L^2.

c. (*4 pts*) Show that $(S_n - a_n)/b_n \to 0$ in probability.

3. (*10 points*) Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\{X_n\}$, $\{Y_n\}$ be real random variables on this probability space.

a. (*5 pts*) Let $\ell : \mathbb{R} \to [0, \infty)$ be a measurable function with compact level sets, i.e. $\{x \in \mathbb{R} : \ell(x) \le c\}$ is a compact set for every $c \ge 0$. Suppose that $\limsup_{n\to\infty} E(\ell(X_n)) < \infty$. Show that $\{X_n\}$ is a tight sequence of random variables.

b. (*5 pts*) Suppose in addition now that Y_n converges to 0 in probability. Show that $X_n Y_n \to 0$ in probability.

4. (*10 points*) Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\{X_n\}$ be independent nonnegative random variables on this probability space. Show that the following are equivalent (i) $\sum_{n=1}^\infty X_n < \infty$ a.s. ; (ii) $\sum_{n=1}^\infty [P(X_n > 1) + E(X_n 1_{X_n \le 1})] < \infty$; $\sum_{n=1}^\infty E(X_n/(1 + X_n)) < \infty$.

[Hint: You may use Kolmogorov's three series theorem as long as you state it precisely.]

5. (*10 points*) Let $(\Omega, \mathcal{F}, P)$ be a probability space.

a. (*5 pts*) Let $\{X_n\}$ be a sequence of real random variables and $\{a_n\}$ be a sequence of positive constants such that $a_n \to \infty$ as $n \to \infty$. Suppose that for some $\theta \in \mathbb{R}$ and a $\mathbb{R}$ valued random variable Z, $a_n(X_n - \theta) \to^d Z$. Let $H : \mathbb{R} \to \mathbb{R}$ be a function that is differentiable at θ with derivative c. Show that

$$a_n(H(X_n) - H(\theta)) \to^d cZ.$$

[Hint: Use Taylor's expansion: For any $x \in \mathbb{R}$, $H(x) = H(\theta) + c(x - \theta) + R(x)(x - \theta)$ where $R(x) \to 0$ as $x \to \theta$.]

b. (*5 pts*) Let $X_1, X_2, \cdots$ be iid $\mathbb{R}^d$ valued random variables with $E(X_1) = \mu \in \mathbb{R}^d$ and finite covariances

$$\Gamma_{ij} = E((X_{n,i} - \mu_i)(X_{n,j} - \mu_j)), \quad i, j = 1, 2, \ldots, d$$

where $X_n = (X_{n,1}, \ldots X_{n,d})'$ and $\mu = (\mu_1, \ldots \mu_d)$. Let $S_n = X_1 + \cdots X_n$. Show that $(S_n - n\mu)/n^{1/2} \to^d \xi$, where ξ is a d-dimensional multivariate normal random variable with mean 0 and covariance Γ.

[Hint: Use Cramér-Wold device. You may use the CLT which was proved in class as long as you state it precisely.]