All problems have equal weight; partial credit will be given for each part of a problem. In budgeting your time, note that some parts will take longer than others. In many cases, different parts of a problem can be done independently, so if you are unsure how to handle one part of a problem, don’t hesitate to try the others. No answer should require a great deal of computation or a complicated proof.

1. Let $\mathbb{R}^{n \times p}$ denote the set of $n \times p$ real matrices, and let $T$ be a bounded subset of $\mathbb{R}^p$. Define a function $f : \mathbb{R}^{n \times p} \to \mathbb{R}$ by

$$f(A) = \inf_{v \in T} ||Av||^2$$

where $||\cdot||$ is the usual Euclidean norm on $\mathbb{R}^n$.

(a) Argue carefully that $\sqrt{f(\cdot)}$ is Lipschitz, taking care to specify the Lipschitz constant, and the appropriate norm.

(b) Let let $U = f(W)$ where $W$ is an $n \times p$ array of i.i.d. standard normal random variables. Find a simple upper bound on $\mathbb{P}(U \geq \mathbb{E}U + 2\sqrt{t \mathbb{E}U} + t)$ for $t > 0$. Hint: Relate the numerical quantity on the right hand side inside the probability to an expression involving $\mathbb{E}\sqrt{U}$ and $\sqrt{t}$.

2. Let $(U_1, V_1), (U_2, V_2), \ldots \in \mathbb{R}^2$ be i.i.d. such that $\mathbb{E}U_i = 1, \mathbb{E}V_i = 2, \text{Var}(U_i) = \text{Var}(V_i) = 1$, and $\text{Cov}(U_i, V_i) = \rho$. What can you say about the limiting behavior of

$$\left( \sum_{i=1}^n U_i \right) \log \left( \frac{\sum_{j=1}^n U_j}{\sum_{k=1}^n V_k} \right) ?$$

3. Let $\mathcal{P} = \{ \mathcal{N}(\mu, \sigma^2) : \mu \in (-\infty, \infty), \sigma^2 > 0 \}$ be the family of univariate normal distributions, and let $X_1, X_2, \ldots$ be i.i.d. with $X_i \sim \mathcal{N}(\mu_0, \sigma_0^2)$.

(a) Find the Fisher information of $\mathcal{P}$.

(b) Find a lower bound on the variance of any unbiased estimate of $\mu_0/\sigma_0$ based on $X_1, \ldots, X_n$. 

(c) What is the maximum likelihood estimate of \( \theta_0 = (\mu_0, \sigma^2_0) \) based on \( X_1, \ldots, X_n \)? (You may simply give the answer – no calculations are necessary.) What can be said about the limiting behavior of this estimate as \( n \) tends to infinity? Briefly justify your answer.

4. Let \( \{U_n\}_{n \geq 1} \) and \( \{V_n\}_{n \geq 1} \) be sequences of random variables in \( \mathbb{R} \) and let \( \{X_n\}_{n \geq 1} \) be a sequence of random vectors in \( \mathbb{R}^d \), with all random quantities being defined on the same probability space. Suppose that \( U_n \to U \) in probability, \( V_n = 2 + o_P(1) \), and \( X_n \Rightarrow X \) in law with \( \mathbb{P}(X = 0) = 0 \). In each of the following cases, (i) indicate whether the quantity converges as \( n \) tends to infinity, (ii) identify the limit if one exists, and (iii) identify the type of convergence. Justify your answers.

(a) \( U_n/V_n \)
(b) \( X_n \log(V_n) \)
(c) \( \mathbb{E}[V_n/(1 + |V_n|)] \)
(d) \( X_n \mathbb{E}[V_n] \)
(e) \( U_n X_n \)
(f) \( ||V_n X_n||^{-1} \)

5. Let \( X \sim \mathcal{N}_d(0, \Sigma_X) \) and \( Y \sim \mathcal{N}_d(0, \Sigma_Y) \) be \( d \)-dimensional multinormal random vectors such that \( \Sigma_Y - \Sigma_X \) is non-negative definite.

(a) Carefully define a random vector \( Z \in \mathbb{R}^d \) with the property that \( X + Z \) and \( X - Z \) have the same distribution as \( Y \).

(b) Let \( C \subseteq \mathbb{R}^d \) be a convex set. Show that \( \mathbb{P}(X \in C^c) \leq 2\mathbb{P}(Y \in C^c) \).