August 2018 Name:______________________

Comprehensive written exam – STOR654 Mathematical Statistics

All problem parts have equal weight. In budgeting your time expect that some part will take longer than others. When solving multi-part problems feel free to use results of earlier parts even if you cannot solve them in proving later parts. You do not have to prove the hint. Do not forget to split the time between both papers!

1. Let $Y_i = e^{X_i}$, $i = 1, \dots, n$, where X_i are i.i.d. $N(\mu, \sigma^2)$.

   (a) Prove that the moment generating function of X_i is
   $$M_{X_i}(t) = e^{\mu t + \sigma^2 t^2/2}.$$

   (b) Find $\theta = EY_i$ and $\operatorname{Var} Y_i$.

   (c) Compute the joint density of $\mathbf{Y} = (Y_1, \dots, Y_n)$. Is it an exponential family?

   (d) Find a minimal sufficient statistic.

   (e) Find $\hat{\theta}_{MM}$ the MM estimator of $\theta = EY_i$ using sample moments of Y_i. What is its MSE?

   (f) Find $\hat{\theta}_{MLE}$ the MLE of θ. What is its MSE?

   (g) Find $\hat{\theta}_U$ the UMVUE of θ. Compare the MSE of $\hat{\theta}_U$ to the MSE of $\hat{\theta}_{MM}$.

   (h) Derive a 95% confidence interval for σ^2.

2. Let X be Negative Binomial(r, p), $r \in \mathbb{N}$, $p \in (0, 1)$. (There are two definitions of negative binomial in the literature. Use the definition from class, i.e., the number of trials for r successes.)

   (a) Prove that for $x > r/p$ the probability
   $$P(X \geq x) \leq p^r (1-p)^{x-r} \left(\frac{x}{r}\right)^r \left(1 - \frac{r}{x}\right)^{r-x}.$$

   Hint: $\sum_{k=r}^{\infty} s^k \binom{k-1}{r-1} = (1-s)^{-r} s^r$.

(b) Consider testing $\mathcal{H}_0 : \ p \geq 0.5$ versus $\mathcal{H}_1 : \ p < 0.5$. Propose a p-value assuming r is known. Let $r = 100$ and assume you observed $x = 318$. Do you reject the null hypothesis at the $\alpha = 10^{-8}$ level?

(c) Let r be fixed and known and consider the Beta(a, b) prior for p. What is the posterior?

(d) Find the Bayes factor for testing $\mathcal{H}_0 : \ p \geq 0.5$ versus $\mathcal{H}_1 : \ p < 0.5$. Evaluate for $r = 100$, $x = 318$, $a = b = 1/2$. (If your calculator cannot evaluate incomplete beta functions, you do not have to enumerate the expressions.)