Read the following information before starting the exam: Show all work, clearly and in order, if you want to get full credit. I reserve the right to take off points if I cannot see how you arrived at your answer. There are 5 problems. Good luck!

1. (10 points) Let \( E = \mathbb{N}_0 \) and let \( \{X_n\}_{n \in \mathbb{N}_0} \) be a Markov chain with transition probability matrix \((p(i,j))_{i,j \in E}\) and probability distributions \(\{P_x\}_{x \in E}\). Suppose for some \( p \in (0,1) \) and \( q = 1-p \), \( p(i,i+1) = p \), \( p(i,i-1) = q \), \( i \geq 1 \). Let \( \sigma_j = \inf\{n \geq 0 : X_n = j\} \) and let for \( s \in [0,1) \), \( \phi(s) = \mathbb{E}_1(s^{\sigma_0}1_{\sigma_0 < \infty}) = \sum_{n=0}^{\infty} s^n P_1(\sigma_0 = n) \).

Show that \( \phi(s) = \frac{1}{2ps} (1 - (1 - 4pq^2s^2)^{1/2}) \), \( s \in (0,1) \).

Using the above result show that \( F(1,0) \equiv P_1(\sigma_0 < \infty) = \begin{cases} 1, & \text{if } p \leq q \\ \frac{q}{p}, & \text{if } p > q. \end{cases} \)

2. (i) (7 points) Let \( \{X_i\}_{i \in I} \) be a uniformly integrable family and \( \{G_j\}_{j \in J} \) be a collection of sub \( \sigma \)-fields of \( \mathcal{F} \). Show that the collection \( U = \{E(X_i \mid G_j), (i,j) \in I \times J\} \) is a u.i. family.

(ii) (3 points) Let \( P \) and \( Q \) be two probability measures on \((\Omega, \mathcal{F})\) such that \( Q \ll P \). Let \( \{G_j, j \in J\} \) be a collection of sub \( \sigma \)-fields of \( \mathcal{F} \). Let \( Q_j \equiv Q \mid G_j \) and \( P_j \equiv P \mid G_j \). Regarding \( Q_j, P_j \) as probability measures on \((\Omega, G_j)\), let \( X_j \) be the \( G_j \) measurable random variable such that \( X_j = \frac{dQ_j}{dP_j} \). Show \( \{X_j, j \in J\} \) is u.i. on \((\Omega, \mathcal{F}, P)\). (You can use part (i) even if you have not proved it).

3. (10 points) Let \( \{X_n\}_{n \geq 0} \) be a submartingale with \( \sup X_n < \infty \) a.s. Let \( \xi_n = X_n - X_{n-1} \) and suppose \( \mathbb{E}(\sup_n \xi_n^+) < \infty \). Show that \( X_n \) converges a.s.

4. (i) (5 points) Show that a collection \( \mathcal{H} \) of functions is uniformly integrable if and only if there is \( H : \mathbb{R}_+ \to \mathbb{R}_+ \) such that \( H(x)/x \to \infty \) as \( x \to \infty \) and \( \sup_{f \in \mathcal{H}} \int H(|f|)d\mu < \infty \).

(ii) (5 points) Let \( \{X_n\}_{n \geq 1} \) be a uniformly integrable \( \mathcal{F}_n \)-martingale. Show that the collection \( \{X_\tau : \tau \text{ is a finite stopping time}\} \) is uniformly integrable. (To answer this part, you can use the optional sampling theorem for bounded stopping times and part (i) without proof).
5. Let $X_1, X_2, \ldots$ be independent random variables with

$$X_n = \begin{cases} 
1, & \text{with probability } \frac{1}{2n}, \\
0, & \text{with probability } 1 - \frac{1}{n}, \\
-1, & \text{with probability } \frac{1}{2n}.
\end{cases}$$

Let $Y_1 = X_1$, and for $n \geq 2$, $Y_n = X_n$ if $Y_{n-1} = 0$, $Y_n = nY_{n-1}|X_n|$ if $Y_{n-1} \neq 0$.

(i) (3 points) Show that $\{Y_n\}_{n \geq 1}$ is a martingale with respect to the filtration $\mathcal{F}_n = \sigma(Y_1, Y_2, \ldots, Y_n)$.

(ii) (2 points) Show that $\{Y_n\}_{n \geq 1}$ converges to zero in probability.

(iii) (5 points) Show that $\{Y_n\}_{n \geq 1}$ does not converge almost surely. Why does the martingale convergence theorem not apply?