STOR 635 CWE (2017-18)

Read the following information before starting the exam: Show all work, clearly and in order, if you want to get full credit. I reserve the right to take off points if I cannot see how you arrived at your answer. There are 5 problems. Good luck!

1. (10 points) Let $E = \mathbb{N}_0$ and let $\{X_n\}_{n\in\mathbb{N}_0}$ be a Markov chain with transition probability matrix $(p(i,j))_{i,j\in E}$ and probability distributions $\{\mathbf{P}_x\}_{x\in E}$. Suppose for some $p \in (0,1)$ and $q = 1-p$, $p(i,i+1) = p$, $p(i,i-1) = q$, $i \geq 1$. Let $\sigma_j \doteq \inf\{n \geq 0 : X_n = j\}$ and let for $s \in [0,1)$,

$$\phi(s) = \mathbf{E}_1(s^{\sigma_0} 1_{\sigma_0 < \infty}) = \sum_{n=0}^{\infty} s^n \mathbf{P}_1(\sigma_0 = n).$$

Show that

$$\phi(s) = \frac{1}{2ps}(1 - (1 - 4pqs^2)^{1/2}),\ s \in (0,1).$$

Using the above result show that

$$F(1,0) \doteq \mathbf{P}_1(\sigma_0 < \infty) = \begin{cases} 1, & \text{if } p \leq q \\ \frac{q}{p}, & \text{if } p > q. \end{cases}$$

2. (i) (7 points) Let $\{X_i\}_{i\in\mathcal{I}}$ be a uniformly integrable family and $\{\mathcal{G}_j\}_{j\in\mathcal{J}}$ be a collection of sub σ-fields of $\mathcal{F}$. Show that the collection $\mathcal{U} \doteq \{\mathbf{E}(X_i \mid \mathcal{G}_j),\ (i,j) \in \mathcal{I}\times\mathcal{J}\}$ is a u.i. family.

(ii) (3 points) Let $\mathbf{P}$ and $\mathbf{Q}$ be two probability measures on $(\Omega, \mathcal{F})$ such that $\mathbf{Q} \ll \mathbf{P}$. Let $\{\mathcal{G}_j,\ j \in \mathcal{J}\}$ be a collection of sub σ-fields of $\mathcal{F}$. Let $\mathbf{Q}_j \doteq \mathbf{Q}\mid_{\mathcal{G}_j}$ and $\mathbf{P}_j \doteq \mathbf{P}\mid_{\mathcal{G}_j}$. Regarding $\mathbf{Q}_j, \mathbf{P}_j$ as probability measures on $(\Omega, \mathcal{G}_j)$, let X_j be the $\mathcal{G}_j$ measurable random variable such that $X_j = \frac{d\mathbf{Q}_j}{d\mathbf{P}_j}$. Show $\{X_j, j \in \mathcal{J}\}$ is u.i. on $(\Omega, \mathcal{F}, \mathbf{P})$. (You can use part (i) even if you have not proved it).

3. (10 points) Let $\{X_n\}_{n\geq 0}$ be a submartingale with $\sup X_n < \infty$ a.s. Let $\xi_n = X_n - X_{n-1}$ and suppose $\mathbb{E}(\sup_n \xi_n^+) < \infty$. Show that X_n converges a.s.

4. (i) (5 points) Show that a collection $\mathcal{H}$ of functions is uniformly integrable if and only if there is $H : \mathbb{R}_+ \to \mathbb{R}_+$ such that $H(x)/x \to \infty$ as $x \to \infty$ and $\sup_{f\in\mathcal{H}} \int H(|f|)d\mu < \infty$.

(ii) (5 points) Let $\{X_n\}_{n\geq 1}$ be a uniformly integrable $\mathcal{F}_n$-martingale. Show that the collection $\{X_\tau : \tau \text{ is a finite stopping time}\}$ is uniformly integrable. (To answer this part, you can use the optional sampling theorem for bounded stopping times and part (i) without proof).

5. Let $X_1, X_2, \ldots$ be independent random variables with

$$X_n = \begin{cases} 1, & \text{with probability } \frac{1}{2n} \\ 0, & \text{with probability } 1 - \frac{1}{n} \\ -1, & \text{with probability } \frac{1}{2n}. \end{cases}$$

Let $Y_1 = X_1$, and for $n \geq 2$, $Y_n = X_n$ if $Y_{n-1} = 0$, $Y_n = nY_{n-1}|X_n|$ if $Y_{n-1} \neq 0$.

(i) (3 points) Show that $\{Y_n\}_{n\geq 1}$ is a martingale with respect to the filtration $\mathcal{F}_n = \sigma(Y_1, Y_2, \ldots, Y_n)$.

(ii) (2 points) Show that $\{Y_n\}_{n\geq 1}$ converges to zero in probability.

(iii) (5 points) Show that $\{Y_n\}_{n\geq 1}$ does not converge almost surely. Why does the martingale convergence theorem not apply?