STOR 634 CWE (2017-18)

Instructions: There are 5 problems. For full credit show all work. Good luck!

P1. Let m be the Lebesgue measure, $A \subset [0,1]$ be measurable, and $0 < \alpha < 1$. Prove that if

$$m(A \cap I) \geq \alpha \cdot m(I) \quad \text{for every interval } I \subset [0,1]$$

then $m(A) = 1$.

P2. Let X_n be a sequence of i.i.d. random variables uniformly distributed on $[0,1]$. Let

$$Y_n = \frac{1}{1 + n^3 X_n^2}.$$

Prove that $\lim_{n\to\infty} Y_n = 0$ almost surely.

P3. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space with $\mu(\Omega) < \infty$. Suppose that $f \in L^1(\mu)$ and

$$a \leq \frac{1}{\mu(A)} \int_A f \, d\mu \leq b$$

for every $A \in \mathcal{F}$ with $\mu(A) > 0$. Prove that $f(\omega) \in [a,b]$ almost surely.

P4. Use characteristic functions to show that there do not exist independent, identically distributed random variables X and Y such that $X - Y$ is distributed uniformly in $[-1,1]$.

P5. Prove the following version of the Central Limit Theorem. Let $X_1, X_2, \ldots$ be independent and uniformly bounded random variables with means 0. Let $S_n = X_1 + \cdots + X_n$. If the variance s_n^2 of S_n goes to infinity then

$$\frac{S_n}{s_n} \to N, \quad \text{in distribution,}$$

where N is a standard normal.