Statistics 655 Comprehensive Written Exam
August 2017

All problems have equal weight; partial credit will be given for each part of a problem. In budgeting your time, note that some parts will take longer than others. In many cases, different parts of a problem can be done independently, so if you are unsure how to handle one part of a problem, don't hesitate to try the others. No answer should require a great deal of computation or a complicated proof.

1. Let $\mathcal{P} = \{f(x : \alpha, \beta) : \alpha, \beta > 0\}$ be a family of densities on $\mathbb{R}^d$ with parameters α and β. Suppose that $\mathcal{P}$ has Fisher information matrix

$$I(\alpha, \beta) \;=\; \begin{bmatrix} 2\alpha^2/\beta & \alpha\beta \\ \alpha\beta & \beta^3 \end{bmatrix}.$$

Let $X_1, X_2, \ldots \in \mathbb{R}^d$ be i.i.d. with $X_i \sim f(x : \alpha, \beta)$. In answering the questions below, you may assume that any necessary regularity and moment conditions hold.

(a) Find a lower bound on the variance of any unbiased estimate of α based on $X_1, \ldots, X_n$ when β is known.

(b) Find a lower bound on the variance of any unbiased estimate of α based on $X_1, \ldots, X_n$ when β is unknown.

(c) Find a lower bound on the variance of any unbiased estimate of α/β based on $X_1, \ldots, X_n$ when α, β are unknown. You may leave your answer in matrix form, but please specify the entries of the matrices.

(c) What can you say about the limiting behavior of the maximum likelihood estimate $\hat{\theta}_n$ of $\theta = (\alpha, \beta)$ based on $X_1, \ldots, X_n$?

2. Let $A \subseteq \mathbb{R}^d$ be a bounded set, and define a function $F : \mathbb{R}^d \to \mathbb{R}$ by

$$F(x) \;=\; \sup_{y \in A} ||x - y||$$

where $|| \cdot ||$ is the standard Euclidean norm.

(a) Let $X \in \mathbb{R}^d$ be a random vector with finite mean. Find an inequality relating $F(\mathbb{E}X)$ and $\mathbb{E}F(X)$. Carefully justify your answer.

(b) Let $X \sim \mathcal{N}_d(\mu, I)$ be a multi-normal random vector with identity variance matrix. Find a bound on the probability that $F(X) - \mathbb{E}F(X)$ is greater than t. Carefully justify your answer.

3. Let $U_1, U_2, \ldots, U$ and $V_1, V_2, \ldots, V$ be random variables defined on the same probability space, such that $U_n \to U$ in probability and $V_n \Rightarrow V$ in law.

(a) Let $g : \mathbb{R} \to \mathbb{R}$ be bounded and uniformly continuous. Does $\mathbb{E}g(U_n V_n) - \mathbb{E}g(U V_n) \to 0$? Justify your answer.

(b) Argue that $U_n V_n \Rightarrow UV$ if U is independent of $V_1, V_2, \ldots, V$.

4. Let $X_1, X_2, \ldots, X \in \mathbb{R}$ be i.i.d. random variables. For $j \geq 1$ define $\mu_j = \mathbb{E}X^j$ to be the jth moment of X if the expectation is well defined.

(a) Show carefully that μ_1 is finite if μ_4 is finite.

(b) Assuming that $\mu_1, \ldots, \mu_4$ are finite, what can you say about the limiting distribution of $\log(1 + \overline{X_n^2})$ where $\overline{X_n^2} = n^{-1} \sum_{i=1}^n X_i^2$?

5. Let $X_1, X_2, \ldots \geq 0$ be identically distributed (not necessarily independent) random variables with finite mean. For $n \geq 1$ define $M_n = \max(X_1, \ldots, X_n)$.

(a) Show that for any $\alpha \geq 0$ we have $M_n \, \mathbb{I}(M_n \leq \alpha) \leq \sum_{i=1}^n X_i \, \mathbb{I}(X_i \leq \alpha)$.

(b) What can you say about the asymptotic behavior of the ratio M_n/n as $n \to \infty$?