August 2017 Name:________________________

COMPREHENSIVE WRITTEN EXAM – STOR654 MATHEMATICAL STATISTICS

Unless otherwise noted, all problem parts have equal weight. In budgeting your time expect that some part will take longer than others. When solving multi-part problems feel free to use results of earlier parts even if you cannot solve them in proving later parts. Do not forget to split the time between both papers.

1. Let us consider a group of n people that play the following game: Each person starts with \$1. Then they people take turns in a pre-assigned order selecting another person at random and taking all the money that person has.

   (a) What is the probability p that no-one steals from you? What does this probability converges to as $n \to \infty$?

   (b) Denote the amount of money at the end of the game owned by the person who went to steal kth as X_k. For example, $X_1 = 2$ with probability p and $X_1 = 0$ with probability $1 - p$. Find EX_k. Which turn would you prefer to be assigned?

   (c) Now consider the turn $k = 1, \ldots, n$ as a parameter. In what follows we will assume that Klára was fortunate enough not to have anything stolen from her and she ended up with \$1. What is the MLE of her turn to steal?

   (d) Find a 95% credible interval using the flat prior for the turn Klára took. (Hint: If general computations become too cumbersome you can instead provide an algorithm and evaluate it on a group of $n = 4$ people.)

2. Let $X_1, X_2, \ldots, X_n$ be i.i.d. with density $f(x) = \lambda e^{-\lambda(x-\beta)} I_{(\beta,\infty)}(x)$, where $\lambda > 0,\ \beta \in \mathbb{R}$ are unknown parameters.

   (a) Is this exponential family? Would the answer change if we assumed $\beta = 0$?

   (b) Find a minimal sufficient statistic.

   (c) Does the distribution of $\frac{X_{(1)} - \beta}{\sum_{i=1}^{n}(X_i - X_{(1)})}$ depend on parameters? What is its density?

(d) Propose a 95% confidence interval for β.

3. Let X be Poisson(λ).

   (a) Prove that for $0 \leq x < \lambda$ the probability $P(X \leq x) \leq \left(\frac{\lambda}{x}\right)^x e^{x-\lambda}$.

   (b) Consider testing $\mathcal{H}_0 : \lambda \geq \lambda_0$ versus $\mathcal{H}_1 : \lambda < \lambda_0$. Propose a p-value. You observed $x = 50$. Do you reject the null hypothesis of $\lambda_0 = 100$ at the $\alpha = 10^{-6}$ level?