STOR 634 Exam: CWE Year: 2016/17

There are 4 questions. Attempt all questions. You may appeal to any result proved in class without proof unless you are specifically asked to "**Give a complete proof.** ". State any result you use. All questions are worth the same number of total points (12.5 points). There is one question with two parts, each part is worth 6.25 points. *Even if you don't know the complete solution, DONT STRESS OUT. Put down your attempt.* Partial credit will be assigned so try your best on each question.

Problem 1. Suppose $(\Omega, \mathscr{F}, \mathbb{P})$ is a probability space and $\{\mathscr{F}_k : k \geq 1\}$ are a sequence of sub-σ fields of $\mathscr{F}$. Show that the sequence $\{\mathscr{F}_k : k \geq 1\}$ is independent **if and only if** each of the pairs $(\sigma(\mathscr{F}_1, \mathscr{F}_2, \ldots, \mathscr{F}_n), \mathscr{F}_{n+1})$ is independent for $n = 1, 2, \ldots$.

Problem 2. Let $\{X_n\}_{n\geq 1}$ be a sequence of iid (independent and identically distributed) random variables and let

$$M_n := \max\left\{|X_j| : 1 \leq j \leq n\right\}.$$

Show that if $\mathbb{E}(|X_1|) < \infty$ then $M_n/n \to 0$ a.s.

Problem 3. Suppose $(\Omega, \mathscr{F})$ is an abstract measure space and let $X : \Omega \to \mathbb{R}_+$ be a (Borel measureable) map. Let $(\mathbb{R}_+, \mathscr{B}(\mathbb{R}_+))$ be the usual Borel space on $\mathbb{R}_+$. Consider the product measureable space,

$$(\Omega \times \mathbb{R}_+, \mathscr{F} \otimes \mathscr{B}(\mathbb{R}_+).$$

Show that the following event A is measureable (i.e. $A \in \mathscr{F} \otimes \mathscr{B}(\mathbb{R}_+)$).

$$A := \{(\omega, t) : t \leq X(\omega)\}$$

Problem 4. Suppose $\{X_n : n \geq 1\}$ is a sequence of independent random variables. Fix $\alpha > 0$. Assume for each $n \geq 1$, X_n is a Bernoulli$(1/n^\alpha)$ random variable, i.e. for each $n \geq 1$,

$$\mathbb{P}(X_n = 1) = \frac{1}{n^\alpha}, \qquad \mathbb{P}(X_n = 0) = 1 - \frac{1}{n^\alpha}.$$

Define $S_n = \sum_{i=1}^n X_i$.

(a) For what values of α does one have that S_n, appropriately re-centered and rescaled, converges in distribution to $N(0,1)$ (i.e. normal with mean zero variance one)?

(b) What happens to S_n for α outside the range you have established in the previous part of the problem? Give a proof of whatever you claim happens in this range.