There are 4 questions. Attempt all questions. You may appeal to any result proved in class without proof unless you are specifically asked to “Give a complete proof.”. State any result you use. All questions are worth the same number of total points (12.5 points). There is one question with two parts, each part is worth 6.25 points. Even if you don’t know the complete solution, DONT STRESS OUT. Put down your attempt. Partial credit will be assigned so try your best on each question.

Problem 1. Suppose $(\Omega, F, P)$ is a probability space and $\{\mathcal{F}_k : k \geq 1\}$ are a sequence of sub-$\sigma$-fields of $F$. Show that the sequence $\{\mathcal{F}_k : k \geq 1\}$ is independent if and only if each of the pairs $(\sigma(\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_n), \mathcal{F}_{n+1})$ is independent for $n = 1, 2, \ldots$.

Problem 2. Let $\{X_n\}_{n \geq 1}$ be a sequence of iid (independent and identically distributed) random variables and let $M_n := \max\{|X_j| : 1 \leq j \leq n\}$. Show that if $E(|X_1|) < \infty$ then $M_n/n \to 0$ a.s.

Problem 3. Suppose $(\Omega, \mathcal{F})$ is an abstract measure space and let $X : \Omega \to \mathbb{R}^+$ be a (Borel measurable) map. Let $(\mathbb{R}^+, \mathcal{B}(\mathbb{R}^+))$ be the usual Borel space on $\mathbb{R}^+$. Consider the product measurable space, $(\Omega \times \mathbb{R}^+, \mathcal{F} \otimes \mathcal{B}(\mathbb{R}^+))$. Show that the following event $A$ is measurable (i.e. $A \in \mathcal{F} \otimes \mathcal{B}(\mathbb{R}^+)$).

$$A := \{(\omega, t) : t \leq X(\omega)\}$$

Problem 4. Suppose $\{X_n : n \geq 1\}$ is a sequence of independent random variables. Fix $\alpha > 0$. Assume for each $n \geq 1$, $X_n$ is a Bernoulli($1/n^\alpha$) random variable, i.e. for each $n \geq 1$,

$$\mathbb{P}(X_n = 1) = \frac{1}{n^\alpha}, \quad \mathbb{P}(X_n = 0) = 1 - \frac{1}{n^\alpha}.$$

Define $S_n = \sum_{i=1}^{n} X_i$.

(a) For what values of $\alpha$ does one have that $S_n$, appropriately re-centered and rescaled, converges in distribution to $N(0, 1)$ (i.e. normal with mean zero variance one)?

(b) What happens to $S_n$ for $\alpha$ outside the range you have established in the previous part of the problem? Give a proof of whatever you claim happens in this range.