Statistics 655 Comprehensive Written Exam
August, 2015

All problems have equal weight; partial credit will be given for each part of a problem. In budgeting your time, note that some parts will take longer than others. In most cases, different parts of a problem can be done independently, so if you are unsure how to handle one part of a problem, don't hesitate to try the others. No answer should require a great deal of computation or a complicated proof.

1. Suppose that $U_n \to U$ in probability, $V_n \to V \geq 0$ in probability and $X_n \Rightarrow X$ in law. Assume each random variable is defined on the same probability space. In each of the following cases, indicate the following: (i) the type of convergence of the sequence (if any) as n tends to infinity; and (ii) the limit of the sequence if it converges, or a counterexample if the sequence may fail to converge. In each case, briefly justify your answer.

   (a) $U_n/(1+V_n)^3$

   (b) U_n/V_n^2

   (c) $U_n + X_n$

   (d) $X_n^2\, e^{X_n}$

   (e) $U_n^2\, e^{V_n}$

   (f) $[X_n + \mathbb{I}(U_n \geq 2)(\log n)^{-1}]\, X_n$

2. Let $X_1, X_2, \ldots \in \mathbb{R}^d$ be i.i.d. random vectors with mean μ and variance matrix Σ.

   (a) Suppose that $\mu \neq 0$. What can you say about the asymptotic distribution of $||\bar{X}_n||$, where $||\cdot||$ denotes the usual Euclidean norm? Justify your answer.

   (b) If there is a limiting distribution in part (a), which mean vector μ will maximize its variance? A short answer is fine; you need not prove anything.

3. Answer the following questions concerning convex sets and functions.

(a) Define what it means for a set $C \subseteq \mathbb{R}^d$ to be convex.

(b) Is the union of two convex sets convex? Justify your answer.

(c) Let $x_1, \ldots, x_n \in \mathbb{R}^d$ and let C be the set of all points of the form $\sum_{i=1}^n \alpha_i \, x_i$ where $\alpha_i \geq 0$ and $\sum_{i=1}^n \alpha_i = 1$. Is C convex? Justify your answer.

(d) Let f be a convex function defined on the set C defined in (c). Show that the maximum of f is achieved at one of the vectors x_i.

(e) Let $X \geq 0$ be a random variable such that $EX^s < \infty$ for each $s \geq 0$. Show that the function $h(s) = \log EX^s$ is convex on $[0, \infty)$.

4. Let $X_1, X_2, \ldots, X \in \mathbb{R}^d$ be random vectors.

(a) Define what is meant by (i) $X_n \Rightarrow X$ (convergence in law), (ii) $X_n = O_P(1)$, and (iii) $X_n \to X$ in probability.

(b) Does $X_n \Rightarrow X$ imply that $X_n = O_P(1)$? Justify your answer.

(c) Does $X_n = O_P(1)$ imply that $X_n \Rightarrow X$ for some random vector X? (Note that the asserted convergence is for the *full* sequence $X_1, X_2, \ldots$.) Briefly justify your answer.

(d) If $X_n \to X$ in probability and $g : \mathbb{R}^d \to \mathbb{R}$ is continuous, does $g(X_n) \to g(X)$ in probability? Justify your answer.

5. Let $\mathcal{P} = \{\mathcal{N}(\mu, \sigma^2) : \mu \in (-\infty, \infty), \, \sigma^2 > 0\}$ be the family of univariate normal distributions, and let $X_1, X_2, \ldots$ be i.i.d. with $X_i \sim \mathcal{N}(\mu_0, \sigma_0^2)$.

(a) Find the Fisher information of $\mathcal{P}$.

(b) Find a lower bound on the variance of any unbiased estimate of μ_0^2 based on $X_1, \ldots, X_n$ when σ_0^2 is unknown.

(c) What can you say about the limiting behavior of the maximum likelihood estimate $\hat{\theta}_n$ of $\theta_0 = (\mu_0, \sigma_0^2)$ based on $X_1, \ldots, X_n$?