CWE 2016: STOR 654 Questions

Note: All parts (7 of them) have an equal weight although their degrees of difficulty vary.

[1] Consider a 3D random vector \((X, Y, Z)\) with mean vector \(\mu\) and covariance matrix
\[
\Sigma = \begin{pmatrix}
1 & \rho & 0 \\
\rho & 1 & \rho \\
0 & \rho & 1
\end{pmatrix}.
\]

1a) Explain what is wrong in the following contradicting statements. When \(\rho = -1\), \(X\) and \(Y\) are colinear that implies a degenerate joint distribution of \((X, Y, Z)\), but \(\Sigma\) is nonsingular.

1b) For what values of \(\rho\) a trivariate normal distribution of \((X, Y, Z)\) can be defined over \(\mathbb{R}^3\)?

1c) Suppose a normal density for \((X, Y, Z)\) is expressed as \(f(x, y, z) = c_0 e^{-Q/2}\) under the condition in (1b) with the quadratic form
\[
Q = c_1(x^2 + y^2 + z^2) - (xy + xz + yz),
\]
where \(c_0, c_1\) are positive constants. Specify the values of \(\rho, c_0\) and \(c_1\).

1d) Show that the density \(f(x, y, z)\) in (1c) would be supported over a subspace of dimension lower than 3 when \(\rho = -1/\sqrt{2}\). Specify the dimensionality of its support.

[2] Useful facts: If \(U \sim beta(a, b)\) — the beta distributions with parameters \(a > 0\) and \(b > 0\), then \(U\) has the density
\[
\pi(u) = \frac{u^{a-1}(1-u)^{b-1}}{B(a, b)}, \ u \in [0, 1],
\]
where \(B(a, b) = \int_0^1 v^{a-1}(1-v)^{b-1} \ dv\), and \(EU = \frac{a}{a+b}\).

Assume \(X\) and \(Y\) are independent random variables with \(X \sim Bernoulli(\lambda)\) and \(Y \sim Bernoulli(\mu)\).
(2a) Let \( \lambda \) and \( \mu \) be iid random variables with the common uniform distribution over the interval \((0,1)\). Denote the prior (joint) density for \((\lambda, \mu)\) by \( \pi \). Consider estimation of the 2D parameter \((\lambda, \mu)\) under the squared error loss \( \| (T_1, T_2) - (\lambda, \mu) \|^2 \) where \( \| \cdot \| \) is the Euclidean distance in \( \mathbb{R}^2 \). Find a Bayesian estimator under the prior \( \pi \) with components \( T_1 \) and \( T_2 \) as functions of \((X,Y)\). [Hint: Notice that \( \| (T_1, T_2) - (\lambda, \mu) \|^2 = |T_1 - \lambda|^2 + |T_2 - \mu|^2 \) and use some simple property of \( \pi \).]

(2b) Explain why it does not make sense to consider Bayesian tests for testing \( H_0 : \lambda = \mu \) vs \( H_1 : \lambda \neq \mu \) with the prior \( \pi \) given in (2a).

(2c) Given \( \beta \in (0,1) \), define a new (mixture) prior \( p = \beta \pi_0 + (1 - \beta) \pi_1 \): on the set \( \Theta_0 = \{ \lambda = \mu \} \), the component of \( p \) is a 1D uniform density \( \pi_0 \) over the interval \((0,1)\); on the set \( \Theta_1 = \{ \lambda \neq \mu \} \), the component of \( p \) is a 2D density \( \pi_1 \) — same as \( \pi \) restricted to \( \Theta_1 \). Find the Bayesian test for \( H_0 \) vs \( H_1 \) under the prior \( p \) and based on the observation \((x,y)\).