

On sublattice determinants in reduced bases

Gábor Pataki and Mustafa Tural *

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Department of Statistics and Operations Research, UNC Chapel Hill

Abstract

Lenstra, Lenstra, and Lovász in [7] proved several inequalities showing that the vectors in an LLL-reduced basis are short, and near orthogonal. Here we present generalizations, from which with $k = 1$, and $k = n$ we can recover their inequalities:

Theorem 1. *Let $b_1, \dots, b_n \in \mathbb{R}^m$ be an LLL-reduced basis of the lattice L , and d_1, \dots, d_k arbitrary linearly independent vectors in L . Then*

$$\|b_1\| \leq 2^{(n-k)/2+(k-1)/4} (\det L(d_1, \dots, d_k))^{1/k}, \quad (1)$$

$$\det L(b_1, \dots, b_k) \leq 2^{k(n-k)/2} \det L(d_1, \dots, d_k), \quad (2)$$

$$\det L(b_1, \dots, b_k) \leq 2^{k(n-k)/4} (\det L)^{k/n}, \quad (3)$$

$$\|b_1\| \cdots \|b_k\| \leq 2^{k(n-k)/2+k(k-1)/4} \det L(d_1, \dots, d_k), \quad (4)$$

$$\|b_1\| \cdots \|b_k\| \leq 2^{k(n-1)/4} (\det L)^{k/n}. \quad (5)$$

□

In the most general setting, we prove:

Theorem 2. *Let $b_1, \dots, b_n \in \mathbb{R}^m$ be an LLL-reduced basis of the lattice L , $1 \leq k \leq j \leq n$, and d_1, \dots, d_j arbitrary linearly independent vectors in L . Then*

$$\det L(b_1, \dots, b_k) \leq 2^{k(n-j)/2+k(j-k)/4} (\det L(d_1, \dots, d_j))^{k/j}, \quad (6)$$

$$\|b_1\| \cdots \|b_k\| \leq 2^{k(n-j)/2+k(j-1)/4} (\det L(d_1, \dots, d_j))^{k/j}. \quad (7)$$

□

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*Department of Statistics and Operations Research, UNC Chapel Hill, gabor@unc.edu, tural@email.unc.edu

1 Lattices and Basis Reduction

A lattice in \mathbb{R}^m is a set of the form

$$L = L(b_1, \dots, b_n) = \left\{ \sum_{i=1}^n \lambda_i b_i \mid \lambda_i \in \mathbb{Z}, (i = 1, \dots, m) \right\}, \quad (8)$$

where b_1, \dots, b_n are linearly independent vectors in \mathbb{R}^m , and are called a *basis* of L . If $B = [b_1, \dots, b_n]$, then we also call B a basis of L , and write $L = L(B)$. The determinant of L is

$$\det L = \sqrt{\det B^T B}, \quad (9)$$

where B is a basis of L , with $\det L$ actually independent of the choice of B .

Finding a short, nonzero vector in a lattice is a fundamental algorithmic problem with many uses in cryptography, optimization, and number theory. For surveys we refer to [2], [3], [11], and [8]. More generally, one may want to find a reduced basis consisting of short, and nearly orthogonal vectors.

A basis b_1, \dots, b_n that is reduced according to the definition of Lenstra, Lenstra, and Lovász [7] is computable in polynomial time in the case of rational lattices, and the b_i are reasonably short, and near orthogonal, namely

$$\|b_1\| \leq 2^{(n-1)/4} (\det L)^{1/n}, \quad (10)$$

$$\|b_1\| \leq 2^{(n-1)/2} \|d\| \text{ for any } d \in L \setminus \{0\}, \quad (11)$$

$$\|b_1\| \cdots \|b_n\| \leq 2^{n(n-1)/4} \det L. \quad (12)$$

hold. Korkhine-Zolotarev (KZ) bases, which were described in [5] by Korkhine, and Zolotarev, and by Kannan in [4] have stronger reducedness properties, for instance, the first vector in a KZ basis is the shortest vector of the lattice. However, KZ bases are computable in polynomial time only when n is fixed. Block KZ bases proposed by Schnorr in [9] form a hierarchy in between: one can trade on the quality of the basis to gain faster computing times.

Our Theorem 1 generalizes inequalities (10) through (12). For instance, (1) with $k = n$ yields (10), and with $k = 1$ yields (11). In turn, from (6) in Theorem 2 with $j = k$, and from (7) with $j = n$ we recover the inequalities of Theorem 1.

It would be interesting to see whether stronger versions of our results can be stated for KZ, or block KZ bases.

As a tool we use Lemma 1 below, which may be of independent interest. For $k = 1$ we can recover from it Lemma (5.3.11) in [2] (proven as part of Proposition (1.11) in [7]). To state it, we will recall the notion of Gram-Schmidt orthogonalization. If $b_1, \dots, b_n \in \mathbb{R}^m$ is a basis of L , then the corresponding Gram-Schmidt vectors b_1^*, \dots, b_n^* , are defined as

$$b_1^* = b_1 \text{ and } b_i^* = b_i - \sum_{j=1}^{i-1} \mu_{ij} b_j \text{ for } i = 1, \dots, n-1, \quad (13)$$

with $\mu_{ij} = \langle b_i, b_j^* \rangle / \langle b_j^*, b_j^* \rangle$, where $\langle \cdot, \cdot \rangle$ is the usual inner product on \mathbb{R}^m .

Lemma 1. *Let d_1, \dots, d_k be linearly independent vectors from the lattice L , and b_1^*, \dots, b_n^* the Gram Schmidt orthogonalization of an arbitrary basis. Then*

$$\det L(d_1, \dots, d_k) \geq \min_{1 \leq i_1 < \dots < i_k \leq n} \{ \|b_{i_1}^*\| \dots \|b_{i_k}^*\| \}. \quad (14)$$

□

In the rest of this section we collect necessary definitions, and results. In Section 2 we prove Lemma 1, and in Section 3 we prove Theorem 2.

We call b_1, \dots, b_n an *LLL-reduced basis* of L , if

$$|\mu_{ji}| \leq 1/2 \quad (j = 2, \dots, n; i = 1, \dots, j-1), \text{ and} \quad (15)$$

$$\|b_j^* + \mu_{j,j-1} b_{j-1}^*\|^2 \geq 3/4 \|b_{j-1}^*\|^2 \quad (1 < j \leq n). \quad (16)$$

From (15) and (16) it follows that

$$\|b_i^*\|^2 \leq 2^{j-i} \|b_j^*\|^2 \quad (1 \leq i \leq j \leq n). \quad (17)$$

If b_1, \dots, b_n are linearly independent vectors, then

$$\det L(b_1, \dots, b_n) = \det L(b_1, \dots, b_{n-1}) \|b'\|, \quad (18)$$

where b' is the projection of b_n on the orthogonal complement of the linear span of b_1, \dots, b_{n-1} .

An integral square matrix U with ± 1 determinant is called unimodular. An elementary column operation performed on a matrix A is either 1) exchanging two columns, 2) multiplying a column by -1 , or 3) adding an integral multiple of a column to another column. Multiplying a matrix A from the right by a unimodular U is equivalent to performing a sequence of elementary column operations on A .

2 Proof of Lemma 1

We need the following

Claim There are elementary column operations performed on d_1, \dots, d_k that yield $\bar{d}_1, \dots, \bar{d}_k$ with

$$\bar{d}_i = \sum_{j=1}^{t_i} \lambda_{ij} b_j \text{ for } i = 1, \dots, k, \quad (19)$$

where $\lambda_{ij} \in \mathbb{Z}$, $\lambda_{i,t_i} \neq 0$, and

$$t_k > t_{k-1} > \dots > t_1. \quad (20)$$

Proof of Claim Let us write

$$BV = [d_1, \dots, d_k], \quad (21)$$

with V an integral matrix. Analogously to how the Hermite Normal Form of an integral matrix is computed, we can do elementary column operations on V to obtain \bar{V} with

$$t_k := \max \{ i \mid \bar{v}_{ik} \neq 0 \} > t_{k-1} := \max \{ i \mid \bar{v}_{i,k-1} \neq 0 \} > \dots > t_1 := \max \{ i \mid \bar{v}_{i1} \neq 0 \}. \quad (22)$$

Performing the same elementary column operations on d_1, \dots, d_k yield $\bar{d}_1, \dots, \bar{d}_k$ which satisfy

$$B\bar{V} = [\bar{d}_1, \dots, \bar{d}_k], \quad (23)$$

so they satisfy (19).

End of proof of Claim

Obviously

$$\det L(\bar{d}_1, \dots, \bar{d}_k) = \det L(d_1, \dots, d_k). \quad (24)$$

Substituting from (13) for b_i we can rewrite (19) as

$$\bar{d}_i = \sum_{j=1}^{t_i} \lambda_{ij}^* b_j^* \text{ for } i = 1, \dots, k, \quad (25)$$

where the λ_{ij}^* are now reals, but $\lambda_{i,t_i}^* = \lambda_{i,t_i}$ nonzero integers.

For all i we have

$$\text{lin} \{ \bar{d}_1, \dots, \bar{d}_{i-1} \} \subseteq \text{lin} \{ b_1^*, \dots, b_{t_{i-1}}^* \}. \quad (26)$$

Therefore

$$\| \text{Proj} \{ \bar{d}_i \mid \{ \bar{d}_1, \dots, \bar{d}_{i-1} \}^\perp \} \| \geq \| \text{Proj} \{ \bar{d}_i \mid \{ b_1^*, \dots, b_{t_{i-1}}^* \}^\perp \} \| \geq \| \lambda_{i,t_i} b_{t_i}^* \| \geq \| b_{t_i}^* \| \quad (27)$$

holds, with the second inequality coming from (20). So applying (18) repeatedly we get

$$\begin{aligned} \det L(\bar{d}_1, \dots, \bar{d}_k) &\geq \det L(\bar{d}_1, \dots, \bar{d}_{k-1}) \| b_{t_k}^* \| \\ &\dots \\ &\geq \| b_{t_1}^* \| \| b_{t_2}^* \| \dots \| b_{t_k}^* \|, \end{aligned} \quad (28)$$

which together with (24) completes the proof. \square

3 Proof of Theorem 1 and Theorem 2

The plan of the proof is as follows: we first prove (1) through (3) in Theorem 1. Then we prove Theorem 2. Finally, (4) follows as a special case of (7) with $j = k$; and (5) as a special case of (7) with $j = n$.

Proof of (1) and (2) Lemma 1 implies

$$\det L(d_1, \dots, d_k) \geq \|b_{t_1}^*\| \|b_{t_2}^*\| \dots \|b_{t_k}^*\| \quad (29)$$

for some $t_1, \dots, t_k \in \{1, \dots, n\}$ distinct indices. Clearly

$$t_1 + \dots + t_k \leq kn - k(k-1)/2 \quad (30)$$

holds. Applying first (17), then (30) yields

$$\begin{aligned} (\det L(d_1, \dots, d_k))^2 &\geq \|b_1^*\|^2 2^{(1-t_1)} \dots \|b_1^*\|^2 2^{(1-t_k)} \\ &= \|b_1^*\|^{2k} 2^{k-(t_1+\dots+t_k)} \\ &\geq \|b_1^*\|^{2k} 2^{k(k+1)/2-kn}, \end{aligned} \quad (31)$$

which is equivalent to (1). Similarly,

$$\begin{aligned} (\det L(d_1, \dots, d_k))^2 &\geq \|b_1^*\|^2 2^{(1-t_1)} \|b_2^*\|^2 2^{(2-t_2)} \dots \|b_k^*\|^2 2^{(k-t_k)} \\ &= \|b_1^*\|^2 \dots \|b_k^*\|^2 2^{(1+\dots+k)-(t_1+\dots+t_k)} \\ &\geq \|b_1^*\|^2 \dots \|b_k^*\|^2 2^{k(k-n)}, \end{aligned} \quad (32)$$

which is equivalent to (2). □

Proof of (3) The proof is by induction. Let us write $D_k = (\det L(b_1, \dots, b_k))^2$. For $k = n-1$, multiplying the inequalities

$$\|b_i^*\|^2 \leq 2^{n-i} \|b_n^*\|^2 \quad (i = 1, \dots, n-1) \quad (33)$$

gives

$$D_{n-1} \leq 2^{n(n-1)/2} (\|b_n^*\|^2)^{n-1} \quad (34)$$

$$= 2^{n(n-1)/2} \left(\frac{D_n}{D_{n-1}} \right)^{n-1}, \quad (35)$$

and after simplifying, we get

$$D_{n-1} \leq 2^{(n-1)/2} (D_n)^{1-1/n}. \quad (36)$$

Suppose that (3) is true for $k \leq n-1$; we will prove it for $k-1$. Since b_1, \dots, b_k forms an LLL-reduced basis of $L(b_1, \dots, b_k)$ we can replace n by k in (36) to get

$$D_{k-1} \leq 2^{(k-1)/2} (D_k)^{(k-1)/k}. \quad (37)$$

By the induction hypothesis,

$$D_k \leq 2^{k(n-k)/2} (D_n)^{k/n}, \quad (38)$$

from which we obtain

$$(D_k)^{(k-1)/k} \leq 2^{(k-1)(n-k)/2} (D_n)^{(k-1)/n}. \quad (39)$$

Using the upper bound on $(D_k)^{(k-1)/k}$ from (39) in (37) yields

$$D_{k-1} \leq 2^{(k-1)/2} 2^{(k-1)(n-k)/2} (D_n)^{(k-1)/k} \quad (40)$$

$$= 2^{(k-1)(n-(k-1))/2} (D_n)^{(k-1)/n}, \quad (41)$$

as required. □

Proof of Theorem 2 From (3) and (2) we have

$$\det L(b_1, \dots, b_k) \leq 2^{k(j-k)/4} (\det L(b_1, \dots, b_j))^{k/j}, \quad (42)$$

$$\det L(b_1, \dots, b_j) \leq 2^{j(n-j)/2} \det L(d_1, \dots, d_j). \quad (43)$$

Raising (43) to the power of k/j gives

$$(\det L(b_1, \dots, b_j))^{k/j} \leq 2^{k(n-j)/2} (\det L(d_1, \dots, d_j))^{k/j}, \quad (44)$$

and plugging (44) into (42) proves (6).

It is shown in [7] that

$$\|b_i\|^2 \leq 2^{i-1} \|b_i^*\|^2 \text{ for } i = 1, \dots, n. \quad (45)$$

Multiplying these inequalities for $i = 1, \dots, k$ yields

$$\|b_1\| \cdots \|b_k\| \leq 2^{k(n-1)/4} \det L(b_1, \dots, b_k), \quad (46)$$

and using (46) with (6) yields (7). □

Remark 1. The k th successive minimum of L is defined as the smallest real number t , such that there are k linearly independent vectors in L with length bounded by t . It is denoted by $\lambda_k(L)$. With the same setup as for (10)-(12) it is shown in [7] that

$$\|b_i\| \leq 2^{n-1} \lambda_i(L) \text{ for } i = 1, \dots, n. \quad (47)$$

For KZ, and block KZ bases similar results were shown in [6], and [10], resp.

The successive minimum results (47) give a more global, and refined view of the lattice, and the reduced basis, than (10) through (12). Our Theorems 1 and 2 are similar in this respect, but they seem to be independent of (47). Of course, multiplying the latter for $i = 1, \dots, k$ gives an upper bound on $\|b_1\| \cdots \|b_k\|$, but in different terms.

The quantities $\det L(b_1, \dots, b_k)$ and $\|b_1\| \dots \|b_k\|$ are also connected by

$$\det L(b_1, \dots, b_k) = \|b_1\| \dots \|b_k\| \sin \theta_2 \dots \sin \theta_k, \quad (48)$$

where θ_i is the angle of b_i with the subspace spanned by b_1, \dots, b_{i-1} . In [1] Babai showed that the sine of the angle of *any* basis vector with the subspace spanned by the other basis vectors in a d -dimensional lattice is at least $(\sqrt{2}/3)^d$. One could combine the lower bounds on $\sin \theta_i$ with the upper bounds on $\det L(b_1, \dots, b_k)$ to find an upper bound on $\|b_1\| \dots \|b_k\|$. However, the result would be weaker than (4) and (5).

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References

- [1] László Babai. On lovász lattice reduction, and the nearest lattice point problem. *Combinatorica*, 6:1–13, 1986.
- [2] Martin Grötschel, László Lovász, and Alexander Schrijver. *Geometric Algorithms and Combinatorial Optimization*, volume 2 of *Algorithms and Combinatorics*. Springer, second corrected edition edition, 1993.
- [3] Ravi Kannan. Algorithmic geometry of numbers. *Annual Review of Computer Science*, 2:231–267, 1987.
- [4] Ravi Kannan. Minkowski’s convex body theorem and integer programming. *Mathematics of Operations Research*, 12(3):415–440, 1987.
- [5] A. Korkine and G. Zolotarev. Sur les formes quadratiques. *Mathematische Annalen*, 6:366–389, 1873.
- [6] Jeffrey C. Lagarias, Hendrik W. Lenstra, and Claus P. Schnorr. Korkine-zolotarev bases and successive minima of a lattice and its reciprocal lattice. *Combinatorica*, 10(4):333–348, 1990.
- [7] Arjen K. Lenstra, Hendrik W. Lenstra, Jr., and László Lovász. Factoring polynomials with rational coefficients. *Mathematische Annalen*, 261:515–534, 1982.
- [8] D. Micciancio. *Complexity of lattice problems: a cryptographic perspective*. Kluwer Academic Publishers, 2002.
- [9] Claus P. Schnorr. A hierarchy of polynomial time lattice basis reduction algorithms. *Theoretical Computer Science*, 53:201–225, 1987.
- [10] Claus P. Schnorr. Block reduced lattice bases and successive minima. *Combinatorics, Probability, and Computing*, 3:507–533, 1994.
- [11] Alexander Schrijver. *Theory of Linear and Integer Programming*. Wiley, Chichester, United Kingdom, 1986.