The Use of Spies in Strategic Situations: Preliminary Report

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Abstract

The use of spies in combat, international, and industrial conflict and competition can be critical in obtaining an advantage in the outcome of such a situation. An understanding of the underlying framework of spying is a valuable tool in making decisions involving the use of spies. This paper studies spying in the context of matrix games. We formulate and model a number of situations involving single or multiple types of spies, spies for one or both sides, and the use of decoy strategies to mislead spies. These all lead to linear programming solutions, which can be solved for fairly large games. We use a simple example throughout to illustrate these situations and their associated solutions.

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1 Introduction

Spies are an important part of many strategic situations such as field combat, international intrigue, and industrial competition. It is not always clear, however, what the value of spies is in competitive situations, and how one decides whether and how to use them. This paper looks at the use of spies in 2-person zero-sum (matrix) games.

The use of spies is part of a larger area of research concerning the value of information in influencing strategy in games. As it is currently studied, this area was originally introduced by Aumann (1974), and has benefitted from extensive subsequent study by Crawford and Sobel (1982), Flesch and Perea y Monsuwé (1999), Forges(1986), Kolberg(1975), Lehrer (1989, 1990, 1991, 1992a,b), Lehrer and Rosenberg (2006), Matsui (1989), and Renault(2000), among others. An explicit treatment of espionage is given by Solan and Yariv (2004), with one-sided use of spies, and including a number of useful — though necessarily small — numerical examples. Most of these papers deal with non-zero sum games with repeated play, where information about the opposing player's payoffs or strategy is gained through signalling by that player. This signal is either incomplete or probabilistic ("noisy"), and hence players only gain partial information about game dynamics. The papers concentrate on determining existence and properties of Nash-type equilibria for these games. The types of games studied are necessarily restricted, usually involving either one-sided signalling or restriction to post-play knowledge of actions, and players have limited or no control over signalling devices. Research has also not focused on finding solutions for these types of games.

This paper takes a more computation-based approach, and considers models where information-gathering is an explicit part of the strategy. We consider single-play zero-sum matrix games, employing mixed strategies. To obtain information players employ an information device called a *probe*, which identifies a restricted subset of strategies that the opposing player will use. The cost of this probe can be represented either as a direct penalty on the payoffs, or indirectly as a loss of strategic choices. These models allow for players to choose from an arsenal of probes, for one or both players to employ probes, and for one or both players to react to the use of probes by using "decoy" strategies to try to fool the probes. The models are solved by an increasingly sophisticated set of linear programs, but can provide solution strategies for fairly large matrix games. We give a simple battlefield-type example to illustrate the various types of models and solutions.

2 The General Setup

This paper will deal exclusively with two person, zero-sum games, described by triple $(A, \mathcal{R}, \mathcal{C})$, where A is the game matrix, $\mathcal{R}$ is the set of m strategies for the row player (denoted "he") and $\mathcal{C}$ is the set of n strategies for the column player (denoted "she"). The entries a_{ij} of the matrix represent the payoff from the column player to the row player when he plays strategy $i \in \mathcal{R}$ and she plays strategy $j \in \mathcal{C}$.

The example we use throughout this paper is a 6×6 *battle game* described as follows: Each player's strategies are identified by a *tactic* and a *location.* A player can choose to either attack or defend, and to do so at one of three possible locations. This results in 6 strategies for each player, with each strategy represented by a pair (T, L) denoting the tactic $T \in \{D, A\}$ and location $L \in \{1, 2, 3\}$. The associated game matrix is given in Table 1. Both players have full knowledge of the game matrix, but do not know what the opposing player's strategy will be.

Normally each player would proceed by finding the optimum *mixed strategy* for the game, the row player's *maximin* strategy denoted by row probability vector $p^0 \in \Re^m$

	$A1$	$A2$	$A3$	$D1$	$D2$	$D3$
$A1$	7	−6	1	−8	−6	−6
$A2$	3	8	−6	−9	−6	9
$A3$	−8	3	−4	4	−4	−3
$D1$	2	−2	7	0	7	−5
$D2$	−3	−3	1	−1	−7	9
$D3$	−6	−4	2	−5	5	−3

Table 1: The Battle Game

		$A1$	$A2$	$A3$	$D1$	$D2$	$D3$	row
	$p_i^0 \setminus q_j^0$	0.277	0.158	0.000	0.425	0.000	0.141	sums
$A1$	0.000	7	−6	1	−8	−6	−6	−3.250
$A2$	0.097	3	8	−6	−9	−6	9	−0.464
$A3$	0.154	−8	3	−4	4	−4	−3	−0.464
$D1$	0.544	2	−2	7	0	7	−5	−0.464
$D2$	0.205	−3	−3	1	−1	−7	9	−0.464
$D3$	0.000	−6	−4	2	−5	5	−3	−4.838
col. sums		−0.464	−0.464	2.817	−0.464	1.178	−0.464	−0.464

Table 2: No-probe solution

for which the minimum entry of the vector $p^0 A$ is maximized[1], and the column player's *minimax* strategy denoted by column probability vector $q^0 \in \Re^n$ for which the maximum entry of the vector Aq^0 is minimized. This results in the classic dual game LPs

$$\begin{aligned} &\max z \\ z &\le \sum_{i\in\mathcal{R}} p_i a_{ij},\ j \in \mathcal{C} \\ \sum_{i\in\mathcal{R}} p_i &= 1 \\ p_i &\ge 0 \end{aligned} \qquad \begin{aligned} &\min w \\ w &\ge \sum_{j\in\mathcal{C}} a_{ij} q_j,\ i \in \mathcal{R} \\ \sum_{j\in\mathcal{C}} q_i &= 1 \\ q_i &\ge 0 \end{aligned}$$

and standard duality theory ensures that these game values will be the same. In the case of our battle game, Table 2 gives the optimal row and column mixed strategies on the left and top, respectively, with the weighted row and column sums on the right and bottom, respectively. These row and column sums have the property that the max row sum and min column sum values, shown in the bottom left-hand corner of the table, are the same value $-.464$. Thus the row player can guarantee a loss of no more than .464, while the column player can guarantee a gain of at least .464, using their optimal mixed strategies, and thus both strategies are optimal.

Now consider the case where either player has the option of gaining information about which strategy the opponent is using. In particular, the row player has the option of employing several *informational probes* $\mathcal{P}_0, \ldots, \mathcal{P}_s$. Each probe $\mathcal{P}_\mu$ is defined by a partition $\mathcal{J}_\mu = (J_1^\mu, \ldots, J_{c_\mu}^\mu)$ of the column space, together with a subset $\mathcal{R}_\mu$ of the row space $\mathcal{R}$. The set of probes could include the *null probe* $\mathcal{P}_0$, with partition $\mathcal{J}_0 = (\mathcal{C})$ and $\mathcal{R}_0 = \mathcal{R}$,

[1]For ease of notation we will use the convention that the row player's strategy as a row vector, and consequently multiplies the game matrix from the right

representing the option of not employing a probe at all. Similar probes can be defined for the column player, consisting of probes $\mathcal{Q}_0, \ldots, \mathcal{Q}_t$, with probe ν associated with restricted column spaces $\mathcal{C}_\nu$ and row space partitions $\mathcal{I}_\nu = (I_1^\nu, \ldots, I_{r_\nu}^\nu)$. In later sections it will be useful to define the sets $J_k^{\mu\nu} = J_k^\mu \cap \mathcal{C}_\nu$ and $I_l^{\nu\mu} = I_l^\nu \cap \mathcal{R}_\mu$, for $\mu = 1, \ldots, s$, $\nu = 1, \ldots, t$, $k = 1, \ldots, c_\mu$, and $l = 1, \ldots, r_\nu$, with $J_k^{\mu\nu}$ representing the strategies in the k^{th} strategy partition set for row probe μ when the column player is employing probe ν, and $I_k^{\nu\mu}$ having the symmetric meaning for the column player.

Each player has the option of using exactly one of the given informational probes. (Since it is not clear how to combine two or more probes, we simply say that using two or more probes is modeled by identifying a third probe that has the appropriate partition and strategy subset so as to model the cost and benefits of combining the probes.) If, say, the row player successfully employs probe $\mathcal{P}_\mu$, then for any strategy $j \in \mathcal{C}$ that the column player uses, the row player is able to determine the strategy partition J_k^μ in which j lies, which we will denote by κ_j^μ. Using probe μ, however, results in his being restricted to strategy subset $\mathcal{R}_\mu$ when he makes his final strategy choice. Probes for the column player work similarly, with λ_i^ν representing the row-strategy partition I_k^ν reported by probe ν when the row player uses strategy i. We note that the functions $\kappa_{(\cdot)}^\mu$ and $\lambda_{(\cdot)}^\nu$ are in fact the "signalling devices" that are mentioned in the literature. In the context of spying, it turns out that it is the *opposing player* that determines which signalling device is used. Both players know the probes available to the other player, and again choose their strategies so as to maximize their gain over any strategy chosen by the opposing player.

For our battle example there will be two nontrivial probes available to each player. The first probe will determine the tactic of the opponent, but using it will restrict the probe-wielding player, in that the player will not be able to attack from Location 2. The second probe will determine the location of the opponent, but using it will mean that the probe-wielding player will not be able to attack from Location 3. Each player can elect not to use any probe, in which case that player has all 6 strategies available, and we will assume that there is no option to use both probes. Using the above notation, the tactic probe $\mathcal{P}_1 = (\mathcal{R}_1, \mathcal{J}_1)$ has

$$\mathcal{R}_1 = \{A1, A3, D1, D2, D3\}$$

$$J_1^1 = \{A1, A2, A3\},\ J_2^1 = \{D1, D2, D3\}$$

and the location probe $\mathcal{P}_2 = (\mathcal{R}_2, \mathcal{J}_2)$ has

$$\mathcal{R}_2 = \{A1, A2, D1, D2, D3\}$$

$$J_1^2 = \{A1, D1\},\ J_2^2 = \{A2, D2\},\ J_3^2 = \{A3, D3\}.$$

As well, both players have access to the null probe $\mathcal{P}_0 = (\mathcal{R}_0, \mathcal{J}_0)$ with $\mathcal{R}_0 = J_1^0 = \{A1, A2, A3, D1, D2, D3\}$.

Using a probe has the potential of improving a player's expected gain. If a probe is successfully employed, then depending upon the report of strategy class given by the probe, the player can assume that he or she is playing the subgame whose columns are restricted to the associated partition set indicated by the probe, but whose columns correspond to that player's restricted strategy class. The amount of advantage for the probe-wielding player is therefore a trade-off between these row and column restrictions. Table 3, together with Table 2, gives the six subgames, together with solutions $p^0, p^A, p^D, p^1, p^2, p^3$ associated with the row player using one of the three probes and receiving the associated reports. Here the use of the location probe guarantees the row player 0 loss, an improvement over not using the probe, even with the loss of strategy $A3$. The use of the tactic probe, however, gives him a loss of .667 due to effect of the loss of the use of strategy $A2$, which is worse than not using any probe. The most effective use of the probes by both

	$p_i^A \setminus q$	$A1$ 0.333	$A2$ 0.667	$A3$ 0.000	row sums
$A1$	0.000	7	−6	1	−1.667
$A3$	0.267	−8	3	−4	−0.667
$D1$	0.733	2	−2	7	−0.667
$D2$	0.000	−3	−3	1	−3.000
$D3$	0.000	−6	−4	2	−4.667
col. sums		−0.667	−0.667	4.067	−0.667

	$p_i^D \setminus q$	$D1$ 0.502	$D2$ 0.231	$D3$ 0.267	row sums
$A1$	0.000	−8	−6	−6	−7.004
$A3$	0.159	4	−4	−3	0.283
$D1$	0.486	0	7	−5	0.283
$D2$	0.355	−1	−7	9	0.283
$D3$	0.000	−5	5	−3	−2.155
col. sums		0.283	0.283	0.283	0.283

Tactic game solutions

	$p_i^1 \setminus q$	$A1$ 0.000	$D1$ 1.000	row sums
$A1$	0.000	7	−8	−8.000
$A2$	0.000	3	−9	−9.000
$D1$	1.000	2	0	0.000
$D2$	0.000	−3	−1	−1.000
$D3$	0.000	−6	−5	−5.000
col. sums		2.000	0.000	0.000

	$p_i^2 \setminus q$	$A2$ 0.565	$D2$ 0.435	row sums
$A1$	0.000	−6	−6	−6.000
$A2$	0.391	8	−6	1.913
$D1$	0.609	−2	7	1.913
$D2$	0.000	−3	−7	−4.739
$D3$	0.000	−4	5	−0.087
col. sums		1.913	1.913	1.913

	$p_i^3 \setminus q$	$A3$ 0.700	$D3$ 0.300	row sums
$A1$	0.000	1	−6	−1.100
$A2$	0.000	−6	9	−1.500
$D1$	0.400	7	−5	3.400
$D2$	0.600	1	9	3.400
$D3$	0.000	2	−3	0.500
col. sums		3.400	3.400	3.400

Location game solutions

Table 3: Row-probe subgames and solution

players, however, requires a more sophisticated method for employing and responding to probes.

The informational probe idea can also be used to model situations where there is no restriction in strategy, but rather where a specified game value penalty is attached to using a probe. In particular, suppose the use of probe $\mathcal{P}_\mu$ imposes a penalty on the row player (which is then "paid" to the column player) of $W_{ij}^\mu \geq 0$ whenever the row player subsequently plays strategy i and the column player plays strategy j. Let W^μ be the resulting matrix of penalties. The probe-partitioned game matrix then looks like

$$\begin{bmatrix} A \\ A - W^1 \\ \vdots \\ A - W^s \end{bmatrix}$$

where the top set I_0 of rows corresponds to the null probe, with $\mathcal{R}_0 = I_0$, and the restricted row space $\mathcal{R}_\mu$ associated with using probe μ is the set of rows corresponding to the subtracted penalty W^μ. For the sake of reasonably-sized games, in this paper we will use the restricted-strategy model rather than the penalty version one.

The remainder of this paper will be devoted to introducing several increasingly more sophisticated uses of probes and decoys in matrix games. Solving these games involves such decisions as the choice of probe, the use of probe information, and the use of decoys, often in the face of an opponent having similar capabilities. We show how we can obtain a complete decision strategy by setting up linear programs that find the required strategy suite for each probe situation.

3 One-sided probes

In this section we treat the case where probes are available only to the row player. The column player knows what probes are available to the row player, but does not know which probe he will be using.

3.1 Single probe

The simplest case is where only one probe $\mathcal{P}$ is available, and must be used. For simplicity we will drop the probe index μ. After using the probe and determining which partition set the column player's strategy is in, the row player picks the appropriate strategy among the restricted row strategies available for that probe. The row player's maximin strategy can be computed as follows: For each probe partition index $k = 1, \ldots, c$, let $\hat{p}^k$ be the maximin row strategy, and $\hat{v}_k$ the game value, associated with the submatrix $A^{\mathcal{R},J_k}$ of A consisting of rows corresponding to $\mathcal{R}$ and columns corresponding to J_k. This is the game the row player is looking at after the probe returns the column strategy partition k. The row player then, by always playing strategy $\hat{p}^k$ whenever the probe returns partition k, can guarantee himself a payoff of

$$v_* = \min_{k=1,\ldots c} \hat{v}_k.$$

The column player's strategy is produced as follows: For each $k = 1, \ldots, c_1$, let $\hat{q}^k$ be the minimax column strategy on the same matrix $A^{\mathcal{R},J_k}$ as above, and so the associated game value is the same $\hat{v}_k$ as the row player. The column player then chooses the minimum index k_* for which $\hat{v}_{k_*} = v_*$ above, and her minimax column strategy is to pick a strategy exclusively from $J^1_{k_*}$ with a mixed strategy of $\hat{q}^{k_*}$. This guarantees her the same value v_* as that of the column player, and so both strategies are optimal for this game.

In our battle game example, suppose that the only probe available is the tactic probe for the row player. Then each player would solve the two tactic-probe subgames as shown in Table 3. The attack-tactic-game has the smallest game value $-.667$. Thus the row player's optimal strategy is to use mixed strategy $p_{A3} = .267$ and $p_{D1} = .733$ if the probe returns the attack partition for the column strategy, and $p_{A3} = .159$, $p_{D1} = .486$, and $p_{D2} = .355$ if the probe returns the defend partition, guaranteeing him a loss of no more than .667. The column player will use as her optimal strategy the play of $q_{A1} = .333$ and $q_{A2} = .667$, not using any defending strategies, guaranteeing her a gain of at least .667, the same as the row player. Thus both strategies are optimal.

3.2 Multiple probes

When multiple probes are available, then the row player must first decide which probe he will employ. If probe μ is used, then after deciding which partition element J^μ_k the column player's strategy is in, he picks the appropriate strategy among the restricted strategies available to him as a result of using that probe.

A simple approach would be for the row player to pick the probe that gives him the best subgame value, in terms of the minimum value over all possible probe reports. In our game, for example, playing optimal strategy in each of the location-probe games guarantees the row player a minimum value of 0, and so this is the probe he would choose. A more sophisticated approach would be to make the choice of probe a mixed strategy. In particular, the row player first determines, for each $k = 1, \ldots, c_\mu$, the maximin mixed strategy $\hat{p}^{\mu k}$ for the game associated with the submatrix $A^{\mathcal{R}_\mu, J^\mu_k}$ of A consisting of rows corresponding to $\mathcal{R}_\mu$ and columns corresponding to J^μ_k. The idea is that row player will choose a probe $\mathcal{P}_\mu$ with probability ρ_μ, and then use the probabilities $\hat{p}^{\mu k}$ to determine the strategy based on the strategy partition J^μ_k returned by the probe. The probabilities ρ_μ are computed by finding the maximin strategy corresponding to the following $s \times n$ game matrix:

$$A^p = \begin{bmatrix} \hat{p}^{1,\kappa^1_1} A^{\mathcal{R}_1,1} & \ldots & \hat{p}^{1,\kappa^1_n} A^{\mathcal{R}_1,n} \\ \vdots & & \vdots \\ \hat{p}^{s,\kappa^s_1} A^{\mathcal{R}_s,1} & \ldots & \hat{p}^{s,\kappa^s_n} A^{\mathcal{R}_s,n} \end{bmatrix}$$

The row player then chooses to use probe μ with probability ρ_μ, and if the probe is used and partition k is observed, then the mixed strategy $\hat{p}^{\mu k}$ is used. For our example, the subgame optimal strategies and row sums are calculated in Tables 2 and 3, and the game matrix is given in Table 4. This gives probe probabilities $\rho_0 = 0$, $\rho_T = .668$, $\rho_L = .332$.

	ρ_i	$A1$	$A2$	$A3$	$D1$	$D2$	$D3$	row sums
Null	0.000	−0.464	−0.464	2.817	−0.464	1.178	−0.464	−0.464
Tactic	0.668	−0.667	−0.667	4.067	0.283	0.283	0.283	0.189
Location	0.332	2.000	1.913	3.400	0.000	1.913	3.400	0.189
col. sums		0.218	0.189	3.846	0.189	0.824	1.317	0.189

Table 4: Multiple probe two-stage solution

Thus the row player's strategy is to use the tactic probe 66.8% of the time and the location probe 33.2% of the time, using the appropriate strategies as given in Table 3 after the probe returns its value. This guarantees the row player a gain of at least .189.

The actual optimal strategy for the row player, however, takes advantage of the fact that in reality, during the subgame play the column player *does not know which probe the row player is using.* To capitalize on this, the row player calculates all of his probe responses against a non-probe-sensitive column strategy. In particular, we need to describe the row player's mixed strategy by a set of probability vectors $\boldsymbol{p}^\mu = (p^{\mu 1}, \ldots, p^{\mu c_\mu})$, $p^{\mu k} \in \Re^{\mathcal{R}_k}$, with $p_i^{\mu k}$ representing the probability that the row player will play probe μ, and if the probe reports back strategy partition k, he will play strategy i. The probability that the row player plays probe μ is $\rho_\mu = \sum_{i \in \mathcal{R}_\mu} p_i^{\mu k}$, which must be the same for all probe response values k. The payoff of this strategy against column player's strategy choice j is now

$$\sum_{\mu=1}^{s} \sum_{i \in \mathcal{R}_\mu} p_i^{\mu \kappa_j^\mu} a_{ij}$$

and so the column player will choose j so as to minimize this value. The row player's maximin strategy, in turn, involves choosing the $\boldsymbol{p}^\mu$'s so as to maximize this minimum value, resulting in the LP:

$$\begin{aligned} \max\ & v \\ v \ \leq\ & \sum_{\mu=1}^{s} \sum_{i \in \mathcal{R}_\mu} p_i^{\mu \kappa_j^\mu} a_{ij} \quad j \in \mathcal{C} \\ \sum_{i \in \mathcal{R}_\mu} p_i^{\mu k} \ =\ & \rho_\mu \quad \mu = 1, \ldots, s,\ k = 1, \ldots, c_\mu \\ \sum_{\mu=1}^{s} \rho_\mu \ =\ & 1 \\ p_i^{\mu k} \geq 0 \quad & \rho_\mu \geq 0. \end{aligned} \tag{1}$$

In our battle example, the solution to the LP in (1) is

	p^0	p^{TA}	p^{TD}	p^{L1}	p^{L2}	p^{L3}
$A1$	0	0	0	0	0	0
$A2$	0	–	–	0	0.2500	0.3633
$A3$	0.1000	0	0.1500	–	–	–
$D1$	0.3571	0.1500	0	0.3929	0.1429	0
$D2$	0	0	0	0	0	0.0296
$D3$	0	0	0	0	0	0
sums	0.4571	0.1500	0.1500	0.3929	0.3929	0.3929

and the associated probe-choice probabilities are $\rho_0 = .4571$, $\rho_T = .1500$, and $\rho_L = .3929$. After normalizing the $p^{\mu k}$ values by dividing by the appropriate ρ_μ values in each subgame, we can recompute the values in Tables 2, 3, and 4, with the new values shown in Table 5. The row player's strategy is to use each probe with probability given by the ρ_μ values, and then use the normalized $\hat{p}^{\mu k}$ values conditioned on which partition the probe reports. This guarantees the row player a game value of at least 1, a significant improvement over the simpler two-stage approach.

The column player's game corresponds to choosing a single probability vector $q \in \Re^{\mathcal{C}}$ that is used regardless of the probe chosen by the row player. Now suppose the row player's chooses probe μ and it returns report of partition class J_k^μ. Then for each choice of a valid row strategy $i \in \mathcal{R}^\mu$, the expected payoff to the row player is

$$\sum_{j \in J_k^\mu} a_{ij} q_j \tag{2}$$

and so the row player's optimal strategy choice for each subgame is the strategy i_* for which $w_{\mu k} = \sum_{j \in J_k^\mu} a_{i_* j} q_j$ is maximized. The total expected gain for each probe choice μ is then the sum of the $w_{\mu k}$, and the row player chooses that probe that gives the maximum payoff. The column strategy, then is to minimize this value, which leads to the LP

$$\begin{aligned} \min \ & w \\ w &\geq \sum_{k=1}^{c_\mu} w_{\mu k} \qquad \mu = 1, \ldots, s, \\ w_{\mu k} &\geq \sum_{j \in J_k^\mu} a_{ij} q_j \quad \mu = 1, \ldots, s, \ k = 1, \ldots, c_\mu, \ i \in \mathcal{R}_\mu \\ \sum_{j \in \mathcal{C}} q_j &= 1 \\ q_j &\geq 0 \end{aligned} \tag{3}$$

This is in fact the dual to (1), obtained by assigning variables q_j, $w_{\mu k}$ and w to each of the constraints in (1), respectively, and thus the two game values will be equal.

In our battle example, the optimal solution to the LP (3) is

$$q_{A1} = .114, \ q_{A2} = 0.228, \ q_{A3} = 0, \ q_{D1} = .482, \ q_{D2} = 0.175, \ q_{D3} = 0.$$

We proceed to normalize these strategies for the tactic and location subgames, in order to more easily compare them to the strategies in Table 3. Specifically, for each partition class J_k^μ we compute the probability $\xi_{\mu k}$ that the column player is playing in partition J_k^μ. These values are given in the last row of Table 5. We now divide the q_j values by the appropriate $\xi_{\mu k}$ values in each subgame, so that $\hat{q}^{J_k^\mu}$ represents the column player's conditional mixed strategy in that subgame. (The values for the no-probe solution remain unchanged.) We recompute the column game values in Tables 2, 3, and 4 with respect

	$p_i^0 \setminus q_j^0$	$A1$ 0.114	$A2$ 0.228	$A3$ 0.000	$D1$ 0.482	$D2$ 0.175	$D3$ 0.000	w_0
$A1$	0.000	7	−6	1	−8	−6	−6	−5.482
$A2$	0.000	3	8	−6	−9	−6	9	−3.228
$A3$	0.219	−8	3	−4	4	−4	−3	1.000
$D1$	0.781	2	−2	7	0	7	−5	1.000
$D2$	0.000	−3	−3	1	−1	−7	9	−2.737
$D3$	0.000	−6	−4	2	−5	5	−3	−3.132
col. sums		−0.188	−0.906	4.594	0.875	4.594	−4.562	1.000

No-probe solution

	$\hat{p}_i^{TA} \setminus \hat{q}^{J_A^T}$	$A1$ 0.333	$A2$ 0.667	$A3$ 0.000	w_{TA}
$A1$	0.000	7	−6	1	−1.667
$A3$	0.000	−8	3	−4	−0.667
$D1$	1.000	2	−2	7	−0.667
$D2$	0.000	−3	−3	1	−3.000
$D3$	0.000	−6	−4	2	−4.667
col. sums		2.000	−2.000	7.000	−0.667

	$\hat{p}_i^{TD} \setminus \hat{q}^{J_D^T}$	$D1$ 0.733	$D2$ 0.267	$D3$ 0.000	w_{TD}
$A1$	0.000	−8	−6	−6	−7.467
$A3$	1.000	4	−4	−3	1.867
$D1$	0.000	0	7	−5	1.867
$D2$	0.000	−1	−7	9	−2.600
$D3$	0.000	−5	5	−3	−2.333
col. sums		4.000	−4.000	−3.000	1.867

Tactic game solutions

	$\hat{p}_i^{L1} \setminus \hat{q}^{J_1^L}$	$A1$ 0.191	$D1$ 0.809	w_{L1}
$A1$	0.000	7	−8	−5.132
$A2$	0.000	3	−9	−6.706
$D1$	1.000	2	0	0.382
$D2$	0.000	−3	−1	−1.382
$D3$	0.000	−6	−5	−5.191
col. sums		2.000	0.000	0.382

	$\hat{p}_i^{L2} \setminus \hat{q}^{J_2^L}$	$A2$ 0.565	$D2$ 0.435	w_{L2}
$A1$	0.000	−6	−6	−6.000
$A2$	0.636	8	−6	1.913
$D1$	0.364	−2	7	1.913
$D2$	0.000	−3	−7	−4.739
$D3$	0.000	−4	5	−0.087
col. sums		4.364	−1.273	1.913

	$\hat{p}_i^{L3} \setminus \hat{q}^{J_3^L}$	$A3$ 0.000	$D3$ 0.000	w_{L3}
$A1$	0.000	1	−6	0.000
$A2$	0.925	−6	9	0.000
$D1$	0.000	7	−5	0.000
$D2$	0.075	1	9	0.000
$D3$	0.000	2	−3	0.000
col. sums		−5.473	9.000	0.000

Location game solutions

	ρ_i	$A1$	$A2$	$A3$	$D1$	$D2$	$D3$
Null	0.457	−0.188	−0.906	4.594	0.875	4.594	−4.562
Tactic	0.150	2.000	−2.000	7.000	4.000	−4.000	−3.000
Location	0.393	2.000	4.364	−5.473	0.000	−1.273	9.000
col. sums		1.000	1.000	1.000	1.000	1.000	1.000

Row player game value computations

Null Probe

strategy played	$D1$
w_0	1.000

Tactic Probe

partition observed	A	D
strategy played	$D1$	$A2$
$\hat{w}_{Tk}$	−0.667	1.867
ξ_{Tk}	0.342	0.658
w_{Tk}	−0.228	1.228
w_T	1.000	

Location Probe

partition observed	1	2	3
strategy played	$D1$	$D1$	$A1$
$\hat{w}_{Lk}$	0.382	1.913	0.000
ξ_{Lk}	0.596	0.404	0.000
w_{Lk}	0.228	0.772	0.000
w_L		1.000	

Column player game value computations

Table 5: One-sided multiprobe optimal solution

to the $\hat{q}^{J_k^\mu}$ values; these are also given in Table 5. Note in these tables that the subgame strategies are generally *not* the optimal strategies for the row or column player, and that the game values are also not the same. The game values given in the lower-right corners of the subgame tables correspond to the column mins, and represent values $\hat{w}_{\mu k}$ for the row player's best play against the column player's $\hat{q}^{J_k^\mu}$ strategy, given that he employed probe μ and that the probe reported column partition J_k^μ. The final sets of values in Table 5 give the corresponding strategy choice for the row player in each subgame, the "re-weighted" game values obtained by multiplying back the subgame probabilities to get the $w_{\mu k}$ values, and finally the sum of the individual probe report values, giving the expected value for each probe choice. The resulting probe game values show that column player loses no more than 1 for any probe choice by the row player, and so both players' strategies are optimal.

3.3 Unreliable probes

The last part of this section deals with the possibility that the chosen probe may fail, in the sense that it gives no information about the column player's strategy. (The situation where the probe may give incorrect information will be discussed in Section 5.) In particular, for each of the row player's probes μ there is a probability γ_μ that the probe is successful. This probability is known to both players. The play proceeds as follows: The row player first chooses probe μ. With probability γ_μ the probe is successful and reports the correct strategy partition for the column player, and the row player chooses his strategy accordingly. With probability $1 - \gamma_\mu$, however, the probe is unsuccessful, and in this case the row player must play his strategy knowing nothing about the column player's strategy. Note that the row player incurs the cost of using the probe whether or not it fails.

To solve this model, we extend the row-player reliable-probe model by using mixed strategy consisting of set $\boldsymbol{p}^\mu = (p^{\mu 1}, \ldots, p^{\mu c_\mu}, p^{\mu F})$ of strategies in $\Re^{\mathcal{R}_k}$, with $p_i^{\mu F}$ representing the probability that he chooses probe μ and, given that is fails, he plays strategy i. The payoff of this strategy against column player's strategy j depends upon whether or not the probe succeeds.

Probe succeeds: This happens with probability γ_μ, and gives expected payoff

$$v_\mu^\gamma = p^{\mu \kappa_j^\mu} A^{\mathcal{R}_\mu, j}$$

Probe fails: This happens with probability $1 - \gamma_\mu$, and gives expected payoff

$$v_\mu^\phi = p^{\mu F} A^{\mathcal{R}_\mu, j}$$

The row player chooses his set of strategies so as to maximize the weighted average $\gamma_\mu v_\mu^\gamma + (1 - \gamma_\mu) v_\mu^\phi$ over all probe choices μ, and so his maximin strategy involves solving the following LP:

$$\begin{array}{lll}
\max \ v & & \\
v & \leq & \displaystyle\sum_{\mu=1}^{s} \left(\gamma_\mu p^{\mu \kappa_j^\mu} A^{\mathcal{R}_\mu, j} + (1 - \gamma_\mu) p^{\mu F} A^{\mathcal{R}_\mu, j} \right) \quad j \in \mathcal{C} \\
\displaystyle\sum_{i \in \mathcal{R}_\mu} p_i^{\mu k} & = & \rho_\mu \qquad\qquad \mu = 1, \ldots, s, \ k = 1, \ldots, c_\mu, F \\
\displaystyle\sum_{\mu=1}^{s} \rho_\mu & = & 1 \\
p_i^{\mu k} \geq 0 & & p_i^{\mu F} \geq 0 \qquad \rho_\mu \geq 0
\end{array} \tag{4}$$

where ρ_μ again is the probability that the row player uses probe μ.

For our battle example, suppose the probe success probabilities are $\gamma_T = .3$ and $\gamma_L = .7$. Since success and failure for the null probe are indistinguishable, we will henceforth arbitrarily assign the null probe success probability 0. The optimal solution to (4) is given in Table 6, with the subgames for each probe combined in the tables. For a particular column strategy choice j, the successful probe-return value represents the use of the row strategy corresponding to the correct partition κ_j^μ of j, the failure value represents the use of the failure strategy, and the γ-weighted average represents the portion of the expectation obtained by the row player upon the column player's strategy choice j. (For simplicity we will no longer be normalizing the probability values for each of the subgames.) The table for the row player value computations sums the portions of the row player expectations over the three probes for each strategy choice j for the row player. This guarantees the row player a game value of at least .990.

The column player's LP is developed along a similar line to that of the LP in (3), again choosing single mixed strategy vector $q \in \Re^{\mathcal{C}}$. Now suppose that the row player has employed probe μ. The portion of the expected payoff to him for each choice of a valid row strategy $i \in \mathcal{R}^\mu$ includes not only the success value $w_{\mu k} = A^{i,J_k^\mu} q_{J_k^\mu}$ of a successful report k, corresponding to (2), but also the failure subgame value $w_{\mu F} = A^{i,\mathcal{C}} q$. The row player chooses a strategy i_k or i_F maximizing each $w_{\mu k}$ or $w_{\mu F}$ separately, and then chooses the probe μ that maximizes the weighted sum

$$w_\mu = \gamma_\mu \sum_{k=1}^{c_\mu} w_{\mu k} + (1 - \gamma_\mu) w_{\mu F}.$$

The column player chooses her strategy q so as to minimize this value, and so her minimax strategy involves solving the following LP:

$$\begin{array}{rcll} \min\ w & & & \\ w & \geq & \gamma_\mu \sum_{k=1}^{c_\mu} w_{\mu k} + (1 - \gamma_\mu) w_{\mu F} & \mu = 1, \ldots, s \\ w_{\mu k} & \geq & A^{i,J_k^\mu} q_{J_k^\mu} & \mu = 1, \ldots, s,\ k = 1, \ldots, c_\mu,\ i \in \mathcal{R}_\mu \\ w_{\mu F} & \geq & A^{i,\mathcal{C}} q & \mu = 1, \ldots, s,\ i \in \mathcal{R}_\mu \\ \sum_{j=1}^{s} q_j & = & 1 & \\ q_j & \geq & 0 & \end{array} \quad (5)$$

This is the dual to (4) as seen by assigning variables q_j, $w_{\mu k}$, including $w_{\mu F}$ and w to the constraints in (4), respectively, and so the strategies of the two players are clearly optimal.

The column player's game values for our battle example are also tabulated in Table 6. The $w_{\mu k}$ values are given on the right in the subgame tables, with the maximizing values underneath. The final set of tables lists, for each probe and report, the row player's strategy choice and resulting expected values. These are summed using the appropriate weight γ_μ or $1 - \gamma_\mu$ to give the value for each probe. The column player can thus also guarantee herself a loss of no more than .990, and so the row and column strategies are optimal.

		A1	A2	A3	D1	D2	D3	
		q						
	p^0	0.146	0.182	0.000	0.514	0.140	0.017	w_0
A1	0.000	7	−6	1	−8	−6	−6	−5.131
A2	0.000	3	8	−6	−9	−6	9	−3.424
A3	0.000	−8	3	−4	4	−4	−3	0.826
D1	0.000	2	−2	7	0	7	−5	0.826
D2	0.000	−3	−3	1	−1	−7	9	−2.331
D3	0.000	−6	−4	2	−5	5	−3	−3.526
col. sums		0.000	0.000	0.000	0.000	0.000	0.000	0.826

No-probe solution

				A1	A2	A3	D1	D2	D3			
				q								
	p^{TA}	p^{TD}	p^{TF}	0.146	0.182	0.000	0.514	0.140	0.017	w_{TA}	w_{TD}	w_{TF}
A1	0.098	0.000	0.000	7	−6	1	−8	−6	−6	−0.073	−5.058	−5.131
A3	0.000	0.439	0.165	−8	3	−4	4	−4	−3	−0.620	1.446	0.826
D1	0.341	0.000	0.274	2	−2	7	0	7	−5	−0.073	0.899	0.826
D2	0.000	0.000	0.000	−3	−3	1	−1	−7	9	−0.985	−1.346	−2.331
D3	0.000	0.000	0.000	−6	−4	2	−5	5	−3	−1.606	−1.921	−3.526
	probe returns A (v_T^γ)			1.368	−1.270	2.487	–	–	–	−0.073	1.446	0.826
	probe returns D (v_T^γ)			–	–	–	1.757	−1.757	−1.318			
	probe fails (v_T^ϕ)			−0.774	−0.052	1.257	0.661	1.257	−1.866			
	γ-weighted average			−0.131	−0.418	1.626	0.990	0.353	−1.701			

Tactic probe solution

					A1	A2	A3	D1	D2	D3				
					q									
	p^{L1}	p^{L2}	p^{L3}	p^{LF}	0.146	0.182	0.000	0.514	0.140	0.017	w_{L1}	w_{L2}	w_{L3}	w_{LF}
A1	0.000	0.000	0.000	0.000	7	−6	1	−8	−6	−6	−3.094	−1.937	−0.101	−5.131
A2	0.000	0.361	0.450	0.000	3	8	−6	−9	−6	9	−4.192	0.618	0.151	−3.424
D1	0.561	0.199	0.000	0.561	2	−2	7	0	7	−5	0.292	0.618	−0.084	0.826
D2	0.000	0.000	0.111	0.000	−3	−3	1	−1	−7	9	−0.952	−1.530	0.151	−2.331
D3	0.000	0.000	0.000	0.000	−6	−4	2	−5	5	−3	−3.448	−0.028	−0.050	−3.526
	probe returns 1 (v_L^γ)				1.121	–	–	0.000	–	–	0.292	0.618	0.151	0.826
	probe returns 2 (v_L^γ)				–	2.491	–	–	−0.772	–				
	probe returns 3 (v_L^γ)				–	–	−2.591	–	–	5.046				
	probe fails (v_L^ϕ)				1.121	−1.121	3.925	0.000	3.925	−2.804				
	γ-weighted average				1.121	1.408	−0.636	0.000	0.637	2.691				

Location probe solution

	A1	A2	A3	D1	D2	D3
Null	0.000	0.000	0.000	0.000	0.000	0.000
Tactic	−0.131	−0.418	1.626	0.990	0.353	−1.701
Location	1.121	1.408	−0.636	0.000	0.637	2.691
col. sums	0.990	0.990	0.990	0.990	0.990	0.990

Row player game value computations

Null Probe

strategy played	D1
w_0	0.826

Tactic Probe

partition observed	A	D	fail
strategy played	D1	A2	D1
subgame values (w_{Tk})	−0.073	1.446	0.826
w_T	0.990		

Location Probe

partition observed	1	2	3	fail
strategy played	D1	D1	D2	D1
subgame values (w_{Tk})	0.292	0.618	0.151	0.826
w_L	0.990			

Column player game value computations

Table 6: One-sided unreliable probe

4 Two-sided probes

When *both* players have access to probes, the situation becomes an interesting modeling problem, since clearly both players cannot know their opponent's strategy while at the same time changing their own as a result of this knowledge. To model this, we will assume that one of the probes obtained information before the other, so that the later probe is correct while the earlier probe is flawed due to the fact that the player with the later information will have changed strategies in accordance with probe information. Although each player knows whether his or her own probe failed, if it does fail that player still does not know whether his opponent's probe succeeded or not. (If a player's probe succeeds, then necessarily the opponent's probe must fail.)

We will deal with the multi-probe case. In particular, for each pair of probes $\mathcal{P}_\mu$ and $\mathcal{Q}_\nu$ we are given probabilities $\gamma_{\mu\nu}$ and $\delta_{\mu\nu}$, $\gamma_{\mu\nu} + \delta_{\mu\nu} \leq 1$, where $\gamma_{\mu\nu}$ is the probability that probe $\mathcal{P}_\mu$ is correct when used against probe $\mathcal{Q}_\nu$, $\delta_{\mu\nu}$ is the probability that probe $\mathcal{Q}_\nu$ is correct when used against probe $\mathcal{P}_\mu$, and $1 - \gamma_{\mu\nu} - \delta_{\mu\nu}$ is the probability that *neither* probe is correct when used against each other. These events are disjoint, and both players know the probabilities $\gamma_{\mu\nu}$ and $\delta_{\mu\nu}$, but not which probe is being used by the opponent.

Again we assume that if a probe fails it gives no information about the opposing player's strategy. Play starts with the row player choosing probe $\mathcal{P}_\mu$ and the column player choosing probe $\mathcal{Q}_\nu$. They await the report of the probes, which either give a strategy partition or fail with probability depending upon the probe choice pair μ and ν. Each player must then set a strategy based on the knowledge they have gained from the probe. The explanation here may seem somewhat out of order; what in fact happens is that the probes come back at different times, so that a successful probe gains its information long enough after its opposing (necessarily failed) probe has reported, that the opposing player has set already his or her final strategy. Thus being able to delay probe deployment as long as possible will have a critical effect on the success of the probe.

To solve this model, we can describe the players' strategies as if they were playing a one-sided unreliable multi-probe game. In particular, mixed strategies will look like $p^\mu = (p^{\mu 1}, \ldots, p^{\mu c_\mu}, p^{\mu F})$, with $p^{\mu k} \in \Re^{\mathcal{R}_\mu}$, $\mu = 1, \ldots, s$ representing the row player's successful and failed strategies, with symmetric strategy $q^\nu = (q^{\nu 1}, \ldots, q^{\nu r_\nu}, q^{\nu F})$, $q^{\nu k} \in \Re^{\mathcal{C}_\nu}$, $\nu = 1, \ldots, t$ for the column player. Let us consider the payoff to the row player against column strategy choice $[\nu, j_F, (j_1, \ldots, j_{r_\nu})]$, where ν is the choice of the column probe, $j_F \in \mathcal{C}_\nu$ is the choice of the failed-probe strategy, and $j_l \in \mathcal{C}_\nu$ is the choice of final strategy if the column probe is successful and reports strategy partition I_l^ν for the row player's strategy.

Row probe correct: This happens with probability $\gamma_{\mu\nu}$, and gives expected payoff

$$v_{\mu\nu}^{\gamma} = p^{\mu \kappa_{j_F}^{\mu}} A^{\mathcal{R}_\mu, j_F}$$

Column probe correct: This happens with probability $\delta_{\mu\nu}$, and gives expected payoff

$$v_{\mu\nu}^{\delta} = \sum_{k=1}^{r_\nu} p_{I_k^{\nu\mu}}^{\mu F} A^{I_k^{\nu\mu}, j_k}$$

Neither probe correct: This happens with probability $1 - \gamma_{\mu\nu} - \delta_{\mu\nu}$, and gives expected payoff

$$v_{\mu\nu}^{\phi} = p^{\mu F} A^{\mathcal{R}_\mu, j_F}.$$

Combining these as we did in the single-probe case, we get the maximin strategy LP for the row player:

$$
\begin{aligned}
&\max\ v \\
v &\le v_{\nu F} + \sum_{k=1}^{r_\nu} v_{\nu k} && \nu = 1,\ldots,t \\
v_{\nu F} &\le \sum_{\mu=1}^{s} \left(\gamma_{\mu\nu} p^{\mu\kappa_j^\mu} A^{\mathcal{R}_\mu, j} + (1-\gamma_{\mu\nu}-\delta_{\mu\nu}) p^{\mu F} A^{\mathcal{R}_\mu, j} \right) && \nu = 1,\ldots,t,\ j \in \mathcal{C}_\nu \\
v_{\nu k} &\le \sum_{\mu=1}^{s} \delta_{\mu\nu} p^{\mu F}_{I_k^{\nu\mu}} A^{I_k^{\nu\mu}, j} && \nu = 1,\ldots,t,\ k = 1,\ldots,r_\nu,\ j \in \mathcal{C}_\nu \\
\sum_{i\in\mathcal{R}_\mu} p_i^{\mu F} &= \rho_\mu && \mu = 1,\ldots,s, \\
\sum_{i\in\mathcal{R}_\mu} p_i^{\mu k} &= \rho_\mu && \mu = 1,\ldots,s,\ k = 1,\ldots,c_\mu \\
\sum_{\mu=1}^{s} \rho_\mu &= 1 \\
p^{\mu F} \ge 0 \qquad & p^{\mu k} \ge 0 \qquad \rho_\mu \ge 0
\end{aligned}
\tag{6}
$$

The dual to (6) is obtained by assigning dual variables ξ_ν, $q_j^{\nu F}$, $q_j^{\nu k}$, $w_{\mu F}$, $w_{\mu k}$, and w to the four constraint sets, respectively, and has the form

$$
\begin{aligned}
&\min\ w \\
w &\ge w_{\mu F} + \sum_{k=1}^{c_\mu} w_{\mu k} && \mu = 1,\ldots,s \\
w_{\mu F} &\ge \sum_{\nu=1}^{t} \delta_{\mu\nu} A^{i,\mathcal{C}_\nu} q^{\nu\lambda_i^\nu} + (1-\gamma_{\mu\nu}-\delta_{\mu\nu}) A^{i,\mathcal{C}_\nu} q^{\nu F} && \mu = 1,\ldots,s,\ i \in \mathcal{R}_\mu \\
w_{\mu k} &\ge \sum_{\nu=1}^{t} \gamma_{\mu\nu} A^{i,J_k^{\mu\nu}} q^{\nu F}_{J_k^{\mu\nu}} && \mu = 1,\ldots,s,\ k = 1,\ldots,c_\mu,\ i \in \mathcal{R}_\mu \\
\sum_{i\in\mathcal{C}_\nu} q_i^{\nu F} &= 1 && \nu = 1,\ldots,t \\
\sum_{i\in\mathcal{C}_\nu} q_i^{\nu k} &= 1 && \nu = 1,\ldots,t,\ k = 1,\ldots,r_\nu \\
\sum_{\nu=1}^{t} \xi_\nu &= 1 \\
q^{\nu F} \ge 0 \qquad & q^{\nu k} \ge 0
\end{aligned}
\tag{7}
$$

Since (7) is symmetric to (6), then the dual variables represent the column player's strategy for this game, and so both game strategies are optimal.

For our battle example, suppose the probe success probabilities $\gamma_{\mu\nu}$ and $\delta_{\mu\nu}$ are as given in Table 7. Table 8 gives the associated optimal LP solution strategies and game value computations for the row player. (The null probe was not used by either player, so we did not display it.) The various scenarios are presented, along with the associated

	Null	Tactic	Location
Null	0	0	0
Tactic	.6	.3	.7
Location	.8	.6	.4

Row Player (γ)

	Null	Tactic	Location
Null	0	.6	1
Tactic	0	.4	.1
Location	0	.3	.5

Column Player (δ)

Table 7: Probe Success Probabilities

v-values. The rows underneath the subgames represent that portion of the expected value corresponding to each of the row probes, allotting strategies to the game values depending upon if the given row probe succeeds (and the column probe fails), if each of the two column probes in turn succeeds and returns a specific report (and the row probe fails), or if both probes fail. These are then collected in the bottom table by *column* probe, multiplying the v^γ values by γ, the v^δ values by δ, and the v^ϕ values by $1-\gamma-\delta$. These are summed and the column player makes her final strategy choice based on the expected gain. The expected returns for each report are summed in turn to give the expected gain from using that probe. This shows that the column player can gain no more than .905 regardless of which probe is chosen.

Table 9 gives the strategies and game value computations for the column player. It is symmetric to Table 8, with w replacing v to denote game values. (The matrices are given in transposed form for ease of presentation, and we will continue doing this for the remainder of the paper.) This shows that for the given column strategies, the row player will lose no more than .905, and so both strategies are optimal.

We should mention one special case that has a considerably simpler solution. This is the case where both players have exactly one probe, and where the single-probe success probabilities γ and δ sum to 1. Thus when a player's probe fails, he or she knows that the opposing player's probe has succeeded. This means in essence that after a player's probe has reported back, that player is playing as if he or she is in the one-sided one-probe situation, on the probe side if the probe comes back positive or on the non-probe side if it comes back negative. The solution to the game is as follows:

1. Solve each of the subgames on $A^{\mathcal{R},J_k}$, $k=1,\ldots,c$, obtaining row strategy $\hat{p}^k$ and column strategy $\tilde{q}^k$, with game value v_k.
2. Solve each of the subgames on $A^{I_l,\mathcal{C}}$, $l=1,\ldots,r$, obtaining column strategy $\hat{q}^l$ and row strategy $\tilde{p}^l$, with game value w_k.
3. If the row player's probe is successful and reports back strategy partition k, then he plays strategy $\hat{p}^k$. If it fails, then he plays strategy $\tilde{p}^{l_*}$ exclusively on partition I_{l_*} for which
$$v_{l_*} = \min_{l=1,\ldots r} v_l.$$
4. If the column player's probe is successful and reports back strategy partition l, then she plays strategy $\hat{q}^l$. If it fails, then she plays strategy $\tilde{p}^{k_*}$ exclusively on partition J_{k_*} for which
$$w_{k_*} = \max_{k=1,\ldots c} w_k.$$

Both players are therefore able to guarantee payoff

$$\gamma v_{l_*} + (1-\gamma) w_{k_*}$$

so that this is the optimal strategy for both players. Note that neither player's strategy is affected by the value of γ.

	p^{TA}	p^{TD}	p^{TF}	$A1$	$A2$	$A3$	$D1$	$D2$	$D3$
$A1$	0.000	0.000	0.000	7	−6	1	−8	−6	−6
$A3$	0.162	0.148	0.000	−8	3	−4	4	−4	−3
$D1$	0.520	0.360	0.529	2	−2	7	0	7	−5
$D2$	0.000	0.174	0.153	−3	−3	1	−1	−7	9
$D3$	0.000	0.000	0.000	−6	−4	2	−5	5	−3
Both Fail $(v^{\phi}_{T0}, v^{\phi}_{TT}, v^{\phi}_{TL})$				0.599	−1.517	3.856	−0.153	2.633	−1.269
Row Success $(v^{\gamma}_{T0}, v^{\gamma}_{TT}, v^{\gamma}_{TL})$				−0.252	−0.556	2.996	0.417	0.705	−0.672
Col. Tactic Success (v^{δ}_{TT})		Reports A		0.000	–	0.000	0.000	0.000	0.000
		Reports D		0.599	–	3.856	−0.153	2.633	−1.269
Col. Location Success (v^{δ}_{TL})		Reports 1		1.058	−1.058	–	0.000	3.703	−2.645
		Reports 2		−0.459	−0.459	–	−0.153	−1.071	1.377
		Reports 3		0.000	0.000	–	0.000	0.000	0.000

Row Tactic Probe

	p^{L1}	p^{L2}	p^{L3}	p^{LF}	$A1$	$A2$	$A3$	$D1$	$D2$	$D3$
$A1$	0.000	0.000	0.000	0.000	7	−6	1	−8	−6	−6
$A2$	0.000	0.318	0.000	0.000	−8	3	−4	4	−4	−3
$D1$	0.318	0.000	0.000	0.318	2	−2	7	0	7	−5
$D2$	0.000	0.000	0.318	0.000	−3	−3	1	−1	−7	9
$D3$	0.000	0.000	0.000	0.000	−6	−4	2	−5	5	−3
Both Fail $(v^{\phi}_{L0}, v^{\phi}_{LT}, v^{\phi}_{LL})$					0.636	−0.636	2.226	0.000	2.226	−1.590
Row Success $(v^{\gamma}_{L0}, v^{\gamma}_{LT}, v^{\gamma}_{LL})$					0.636	2.544	0.318	0.000	−1.908	2.862
Col. Tactic Success (v^{δ}_{LT})			Reports A		0.000	–	0.000	0.000	0.000	0.000
			Reports D		0.636	–	2.226	0.000	2.226	−1.590
Col. Location Success (v^{δ}_{LL})			Reports 1		0.636	−0.636	–	0.000	2.226	−1.590
			Reports 2		0.000	0.000	–	0.000	0.000	0.000
			Reports 3		0.000	0.000	–	0.000	0.000	0.000

Row Location Probe

Column Probe	Outcome	$A1$	$A2$	$A3$	$D1$	$D2$	$D3$	Strategy	Strategy value	Probe value
Null	Both Fail	0.367	−0.734	1.988	−0.061	1.498	−0.825			
	Row Success	0.358	1.702	2.052	0.250	−1.103	1.886			
	Total Col. Failure	0.724	0.968	4.040	0.189	0.395	1.061	$D1$	0.189	0.189
Tactic	Both Fail	0.243	–	1.379	−0.046	1.012	−0.540			
	Row Success	0.306	–	1.090	0.125	−0.933	1.515			
	Total Col. Failure	0.549	–	2.469	0.079	0.079	0.976	$D2$	0.079	
	Reports A	0.000	–	0.000	0.000	0.000	0.000	$D3$	0.000	
	Reports D	0.430	–	2.210	−0.061	1.721	−0.984	$D3$	−0.984	−0.905
Location	Both Fail	0.183	−0.367	–	−0.031	0.749	−0.413			
	Row Success	0.078	0.628	–	0.292	−0.269	0.674			
	Total Col. Failure	0.261	0.261	–	0.261	0.480	0.261	$D3$	0.261	
	Reports 1	0.424	−0.424	–	0.000	1.483	−1.059	$D3$	−1.059	
	Reports 2	−0.046	−0.046	–	−0.015	−0.107	0.138	$D2$	−0.107	
	Reports 3	0.000	0.000	–	0.000	0.000	0.000	$A1$	0.000	−0.905

Summary Values

Table 8: Two-sided probe: Row player's value

	q^{TA}	q^{TD}	q^{TF}	$A1$	$A2$	$A3$	$D1$	$D2$	$D3$
$A1$	0.088	0.000	0.000	7	3	−8	2	−3	−6
$A3$	0.000	0.000	0.000	1	−6	−4	7	1	2
$D1$	0.065	0.000	0.033	−8	−9	4	0	−1	−5
$D2$	0.000	0.000	0.121	−6	−6	−4	7	−7	5
$D3$	0.000	0.154	0.000	−6	9	−3	−5	9	−3
Both Fail ($w^{\phi}_{T0}, w^{\phi}_{TT}, w^{\phi}_{TL}$)				−0.987	−1.020	−0.352	0.845	−0.878	0.440
Column Success ($w^{\delta}_{T0}, w^{\delta}_{TT}, w^{\delta}_{TL}$)				0.092	−0.325	−0.442	−0.768	1.382	−0.461
Row Tactic Success (w^{γ}_{TT})		Reports A		0.000	–	0.000	0.000	0.000	0.000
		Reports D		−0.296	–	−0.106	0.254	−0.263	0.132
Row Location Success (w^{γ}_{TL})		Reports 1		−0.157	−0.177	–	0.000	−0.020	−0.098
		Reports 2		−0.435	−0.435	–	0.507	−0.507	0.362
		Reports 3		0.000	0.000	–	0.000	0.000	0.000

Column Tactic Probe

	q^{L1}	q^{L2}	q^{L3}	q^{LF}	$A1$	$A2$	$A3$	$D1$	$D2$	$D3$
$A1$	0.000	0.000	0.846	0.229	7	3	−8	2	−3	−6
$A2$	0.000	0.000	0.000	0.459	1	−6	−4	7	1	2
$D1$	0.000	0.000	0.000	0.098	−8	−9	4	0	−1	−5
$D2$	0.000	0.846	0.000	0.000	−6	−6	−4	7	−7	5
$D3$	0.846	0.000	0.000	0.060	−6	9	−3	−5	9	−3
Both Fail ($w^{\phi}_{L0}, w^{\phi}_{LT}, w^{\phi}_{LL}$)					−2.293	4.012	−0.245	−0.758	−1.625	−3.883
Column Success ($w^{\delta}_{L0}, w^{\delta}_{LT}, w^{\delta}_{LL}$)					−5.079	−5.079	−6.772	−4.232	−5.925	−5.079
Row Tactic Success (w^{γ}_{LT})			Reports A		−0.803	–	−0.321	−0.321	−1.446	−2.249
			Reports D		−1.098	–	0.044	0.044	0.044	−0.338
Row Location Success (w^{γ}_{LL})			Reports 1		0.170	−0.256	–	0.184	−0.334	−0.846
			Reports 2		−1.536	1.034	–	0.140	−1.058	−0.372
			Reports 3		−0.143	0.215	–	−0.120	0.215	−0.072

Column Location Probe

Column Probe	Outcome	$A1$	$A2$	$A3$	$D1$	$D2$	$D3$	Strategy	Strategy value	Probe value
Null	Both Fail	−0.395	−0.408	−0.141	0.338	−0.351	0.176			
	Column Success	−5.023	−5.274	−7.037	−4.693	−5.096	−5.355			
	Total Row Failure	−5.418	−5.682	−7.178	−4.355	−5.447	−5.179	$D1$	−4.355	−4.355
Tactic	Both Fail	−0.755	–	−0.154	0.102	−0.588	−0.645			
	Column Success	−0.471	–	−0.854	−0.730	−0.040	−0.692			
	Total Row Failure	−1.226	–	−1.009	−0.628	−0.628	−1.337	$D2$	−0.628	
	Reports A	−0.803	–	−0.321	−0.321	−1.446	−2.249	$A3$	−0.321	
	Reports D	−1.098	–	0.044	0.044	0.044	−0.338	$A3$	0.044	−0.905
Location	Both Fail	−0.328	0.299	–	0.009	−0.250	−0.344			
	Column Success	−2.512	−2.637	–	−2.346	−2.548	−2.678			
	Total Row Failure	−2.840	−2.338	–	−2.338	−2.798	−3.022	$A2$	−2.338	
	Reports 1	0.170	−0.256	–	0.184	−0.334	−0.846	$D1$	0.184	
	Reports 2	−1.536	1.034	–	0.140	−1.058	−0.372	$A2$	1.034	
	Reports 3	−0.143	0.215	–	−0.120	0.215	−0.072	$A2$	0.215	−0.905

Summary Values

Table 9: Two-sided probe: Column player's value

In our battle example, suppose that the row player uses the tactic probe, the column player uses the location probe, and the associated probe success probabilities are $\gamma = .7$ and $\delta = .3$, respectively. Table 10 gives the solutions to the five subgames. From this we get that if the row player's tactic probe succeeds, then he uses mixed strategy $\hat{p}^A$ or $\hat{p}^D$ given in the table, depending upon the report, and the column player uses strategy $\hat{q}^A$, *i.e.*, $q_{A1} = .333$ and $q_{A2} = .667$. If the row player's tactic probe fails, then the column player's location probe will succeed, and so she uses strategy $\hat{q}^1$ $(D3)$, $\hat{q}^2$ $(D2)$, or $\hat{q}^3$ $(A1)$ depending upon the report, and the column player uses strategy $\tilde{p}^1$ $(D1)$. This guarantees both players a game value of

$$\gamma v_D + (1-\gamma)w_1 = .7 \times (-.667) + .3 \times (-5) = -1.967$$

5 The use of decoys

As a final twist to the use of probes, we allow the player being probed to have the option of setting up a *decoy strategy*, representing the strategy the probe sees *before* the opposing player sets his or her true strategy. Knowing that a decoy strategy could be employed will obviously have a significant effect on how each player responds in setting his or her final strategy.

5.1 One-sided probes with decoys

Let us begin with the case where the row player is the only player with probes available to him, but where the column player can use decoy strategies. The game proceeds as follows: At the same time the row player chooses a probe μ, the column player independently chooses a decoy strategy d, with the intent to switch to her true strategy j at the last possible moment. The row player likewise sends his probe out at the last possible moment. Now for each probe choice μ there is probability γ_μ that the row player sees the column player's true strategy partition κ_j^μ, and probability $1-\gamma_\mu$ that he sees her decoy strategy partition κ_d^μ. Neither player knows which type of strategy the row player has seen, but both players know the probability γ_μ that the row player will see the true strategy partition when employing probe μ.

As in the non-decoy case, the mixed strategies for the row player are described by set $\boldsymbol{p}^\mu = (p^{\mu 1}, \ldots, p^{\mu c_\mu})$, $p^{\mu k} \in \mathcal{R}_\mu$, of vectors, where $p_i^{\mu k}$ is the probability that the row player will employ probe μ and, upon seeing strategy partition k from the column player, will use true strategy i. Again, $\rho_\mu = \sum_{i \in \mathcal{R}_\mu} p_i^{\mu k}$ is the probability that probe μ is used, common over all k. We now compute the expected gain for the column player when she uses pure strategy consisting of decoy strategy d and final strategy j.

Row probe correct: With probability γ_μ the row player will be seeing the column player's correct strategy, and in this case the expected payoff will be

$$v_\mu^\gamma = p^{\mu\kappa_j^\mu} A^{\mathcal{R}_\mu, j}.$$

Row probe incorrect: With probability $1-\gamma_\mu$ the row player will be seeing the column player's decoy strategy d, and so the expected payoff will be

$$v_\mu^\phi = p^{\mu\kappa_d^\mu} A^{\mathcal{R}_\mu, j}.$$

	$\hat{p}^A \setminus \tilde{q}^A$	$A1$ 0.333	$A2$ 0.667	row sums/ v_A
$A1$	0.000	7	−6	−1.667
$A3$	0.267	−8	3	−0.667
$D1$	0.733	2	−2	−0.667
$D2$	0.000	−3	−3	−3.000
$D3$	0.000	−6	−4	−4.667
col. sums		−0.667	−0.667	−0.667

	$\hat{p}^D \setminus \tilde{q}^D$	$D1$ 0.502	$D2$ 0.231	$D3$ 0.267	row sums/ v_D
$A1$	0.000	−8	−6	−6	−7.004
$A3$	0.159	4	−4	−3	0.283
$D1$	0.486	0	7	−5	0.283
$D2$	0.355	−1	−7	9	0.283
$D3$	0.000	−5	5	−3	−2.155
col. sums		0.283	0.283	0.283	0.283

Row probe-success solutions

	$\tilde{p}^1 \setminus \hat{q}^1$	$A1$ 0.000	$A2$ 0.000	$D1$ 0.000	$D2$ 0.000	$D3$ 1.000	row sums/ w_1
$A1$	0.000	7	−6	−8	−6	−6	−6.000
$D1$	1.000	2	−2	0	7	−5	−5.000
col. sums		2.000	−2.000	0.000	7.000	−5.000	−5.000

	$\tilde{p}^2 \setminus \hat{q}^2$	$A1$ 0.000	$A2$ 0.000	$D1$ 0.000	$D2$ 1.000	$D3$ 0.000	row sums/ w_2
$D2$	1.000	−3	−3	−1	−7	9	−7.000
col. sums		−3.000	−3.000	−1.000	−7.000	9.000	−7.000

	$\tilde{p}^3 \setminus \hat{q}^3$	$A1$ 1.000	$A2$ 0.000	$D1$ 0.000	$D2$ 0.000	$D3$ 0.000	row sums/ w_3
$A3$	0.000	−8	3	4	−4	−3	−8.000
$D3$	1.000	−6	−4	−5	5	−3	−6.000
col. sums		−6.000	−4.000	−5.000	5.000	−3.000	−6.000

Column probe-success solutions

Table 10: Two-side probe: special case

Thus the column player's best strategy is to choose decoy strategy d and true strategy j that minimizes the sum of $\gamma v_\mu^\gamma + (1-\gamma_\mu)v_\mu^\phi$ over all probes μ, which leads to the LP

$$
\begin{aligned}
\max\ & v \\
v &\leq \sum_{\mu=1}^{s} \left(\gamma_\mu p^{\mu\kappa_j^\mu} A^{\mathcal{R}_\mu,j} + (1-\gamma_\mu) p^{\mu\kappa_d^\mu} A^{\mathcal{R}_\mu,j}\right) && d,j \in \mathcal{C} \\
\sum_{i\in\mathcal{R}_\mu} p_i^{\mu k} &= \rho_\mu && \mu = 1,\ldots,s,\ k=1,\ldots,c_\mu \\
\sum_{\mu=1}^{s} \rho_\mu &= 1 \\
p_l^{\mu k} &\geq 0
\end{aligned}
\tag{8}
$$

For our battle example, suppose the probe success probabilities are $\gamma_T = .6$ and $\gamma_L = .8$. (Again, whether the null probe succeeds is immaterial.) Table 11 gives the associated optimal LP solution strategies and game value computations for the row player. If the probe is correct the row player will be playing the mixed strategy corresponding to the partition class of that particular column, and if it is incorrect he will be always be playing the mixed strategy for the decoy partition class. The table groups the expected gains for an incorrect probe by partition class of the decoy, and the expected gain value is the sum of the successful and unsuccessful probe values weighted by γ_μ and $1-\gamma_\mu$, respectively. These are in turn summed for each specific decoy strategy in the summary table, and this shows that regardless of the decoy chosen the row player can guarantee himself a game value of at least .318.

The column player has no access to probes, but she does have the opportunity to set a decoy. Her strategy, then, is comprised of collection $\boldsymbol{q} = (q^d)$, $d \in \mathcal{C}$, where q^d is the mixed strategy used by the column player after presenting decoy $d \in \mathcal{C}$. The payoff to the row player of this strategy, when he chooses row probe μ and response strategy $(i_1,\ldots,i_{c_\mu})$, and the probe subsequently returns value k, again depends upon the success of his probe.

Row probe correct: With probability γ_μ the row player sees the actual column player's strategy, and so his expected payoff is

$$w_\mu^\gamma = \sum_{d\in\mathcal{C}} A^{i_k,J_k^\mu} q^d_{J_k^\mu}$$

Row probe incorrect: With probability $1-\gamma_\mu$ the row player sees the column player's decoy strategy, and so his expected payoff is

$$w_\mu^\phi = \sum_{d\in J_k^\mu} A^{i_k,\mathcal{C}} q^d$$

	p^0	$A1$	$A2$	$A3$	$D1$	$D2$	$D3$
$A1$	0.000	7	−6	1	−8	−6	−6
$A2$	0.000	3	8	−6	−9	−6	9
$A3$	0.000	−8	3	−4	4	−4	−3
$D1$	0.291	2	−2	7	0	7	−5
$D2$	0.000	−3	−3	1	−1	−7	9
$D3$	0.000	−6	−4	2	−5	5	−3
v_0^ϕ		0.582	−0.582	2.038	0.000	2.038	−1.456

Row Null Probe

		p^{TA}	p^{TD}	$A1$	$A2$	$A3$	$D1$	$D2$	$D3$
	$A1$	0.000	0.000	7	−6	1	−8	−6	−6
	$A3$	0.072	0.268	−8	3	−4	4	−4	−3
	$D1$	0.196	0.000	2	−2	7	0	7	−5
	$D2$	0.000	0.000	−3	−3	1	−1	−7	9
	$D3$	0.000	0.000	−6	−4	2	−5	5	−3
	Probe correct (v_T^γ)			−0.186	−0.175	1.081	1.072	−1.072	−0.804
Decoy A1,A2,A3	Probe incorrect (v_T^ϕ)			−0.186	−0.175	1.081	0.289	1.081	−1.195
	Expected gain			−0.186	−0.175	1.081	0.759	−0.211	−0.960
Decoy D1,D2,D3	Probe incorrect (v_T^ϕ)			−2.143	0.804	−1.072	1.072	−1.072	−0.804
	Expected gain			−0.969	0.217	0.220	1.072	−1.072	−0.804

Row Tactic Probe

		p^{L1}	p^{L1}	p^{L3}	$A1$	$A2$	$A3$	$D1$	$D2$	$D3$
	$A1$	0.000	0.000	0.000	7	−6	1	−8	−6	−6
	$A3$	0.000	0.245	0.220	−8	3	−4	4	−4	−3
	$D1$	0.441	0.196	0.000	2	−2	7	0	7	−5
	$D2$	0.000	0.000	0.221	−3	−3	1	−1	−7	9
	$D3$	0.000	0.000	0.000	−6	−4	2	−5	5	−3
	Probe correct (v_L^γ)				0.882	1.564	−1.100	−0.000	−0.093	3.969
Decoy A1,D1	Probe incorrect (v_L^ϕ)				0.882	−0.882	3.087	−0.000	3.087	−2.205
	Expected gain				0.882	1.075	−0.262	−0.000	0.543	2.734
Decoy A2,D2	Probe incorrect (v_L^ϕ)				−0.002	1.098	−1.100	−2.202	−2.867	3.969
	Expected gain				0.705	1.471	−1.100	−0.440	−0.648	3.969
Decoy A3,D3	Probe incorrect (v_L^ϕ)				−0.002	1.098	−1.100	−2.202	−2.867	3.969
	Expected gain				0.705	1.471	−1.100	−0.440	−0.648	3.969

Row Location Probe

Decoy	$A1$	$A2$	$A3$	$D1$	$D2$	$D3$	Strategy	Value
$A1$	1.278	0.318	2.856	0.759	2.370	0.318	$A2$	0.318
$A2$	1.101	0.714	2.019	0.318	1.179	1.553	$D1$	0.318
$A3$	1.101	0.714	2.019	0.318	1.179	1.553	$D1$	0.318
$D1$	0.495	0.710	1.995	1.072	1.509	0.475	$D3$	0.475
$D2$	0.318	1.106	1.158	0.631	0.318	1.709	$D2$	0.318
$D3$	0.318	1.106	1.158	0.631	0.318	1.709	$D2$	0.318

Summary Values

Table 11: One-sided probe with decoys: row player's value

Thus the row player's best strategy is to choose probe μ that minimizes the sum of $\gamma_\mu v_\mu^\gamma + (1-\gamma_\mu)v_\mu^\phi$ over all probes μ and, for each response $k = 1,\ldots,c_\mu$, over every choice of final strategy $i \in \mathcal{R}_\mu$. This leads to LP

$$
\begin{aligned}
&\min w \\
&\quad w \;\geq\; \sum_{k=1}^{c_\mu} w_{\mu k} && \mu = 1,\ldots,s \\
&\quad w_{\mu k} \;\geq\; \gamma_\mu \sum_{d\in\mathcal{C}} A^{i,J_k^\mu} q^d_{J_k^\mu} + (1-\gamma_\mu) \sum_{d\in J_k^\mu} A^{i,\mathcal{C}} q^d && \mu = 1,\ldots,s,\; k=1,\ldots,c_\mu,\; i\in\mathcal{R}_\mu \\
&\sum_{d,j\in\mathcal{C}} q_j^d \;=\; 1 \\
&\quad q_j^d \;\geq\; 0.
\end{aligned}
\tag{9}
$$

This is the dual LP to (8), where the dual variables q_j^d, $w_{\mu k}$ and w are attached to the three constraints of (8), respectively. Thus by solving (18) the column player obtains her optimal strategy, and hence the two game values are equal.

For our battle game, Table 12 gives the associated optimal LP solution strategies and game value computations for the column player. The expected values for each probe and each report are given, and the expected gain is again the weighted sum of the successful and unsuccessful values. The row player chooses the best strategy for each report, and for each probe his gain is the sum of the gains for each report. Again this shows that regardless of the probe chosen, the column player can guarantee herself a loss of no more than .318, and so the two strategies are optimal.

	q^{A1}	q^{A2}	q^{A3}	q^{D1}	q^{D2}	q^{D3}	$A1$	$A2$	$A3$	$D1$	$D2$	$D3$	Strategy	Strategy value	Probe value
$A1$	0.000	0.000	0.000	0.000	0.000	0.255	7	3	−8	2	−3	−6			
$A2$	0.123	0.000	0.000	0.000	0.000	0.000	−6	8	3	−2	−3	−4			
$A3$	0.000	0.000	0.000	0.000	0.000	0.000	1	−6	−4	7	1	2			
$D1$	0.000	0.360	0.195	0.000	0.000	0.000	−8	−9	4	0	−1	−5			
$D2$	0.000	0.000	0.000	0.000	0.000	0.032	−6	−6	−4	7	−7	5			
$D3$	0.034	0.000	0.000	0.000	0.000	0.000	−6	9	−3	−5	9	−3			
Null Probe	w_0^ϕ						−3.795	−3.136	0.318	0.318	−1.606	−4.742	$D1$		0.318
Tactic Probe	Correctly reports A (w_T^γ)						1.049	–	−1.672	0.264	−1.134	−2.022			
	Incorrectly reports A (w_T^ϕ)						−5.387	–	2.488	−0.417	−0.615	−3.372			
	Expected gain						−1.526	–	−0.008	−0.008	−0.927	−2.562	$A3$	−0.008	
	Correctly reports D (w_T^γ)						−4.843	–	1.990	0.054	−0.472	−2.720			
	Incorrectly reports D (w_T^ϕ)						1.593	–	−2.170	0.735	−0.991	−1.370			
	Expected gain						−2.269	–	0.326	0.326	−0.679	−2.180	$D1$	0.326	0.318
Location Probe	Correctly reports 1 (w_L^γ)						−2.658	−4.234	–	0.510	−1.321	−4.308			
	Incorrectly reports 1 (w_L^ϕ)						−0.943	1.292	–	−0.417	−0.060	−0.594			
	Expected gain						−2.315	−3.129	–	0.325	−1.069	−3.566	$D1$	0.325	
	Correctly reports 2 (w_L^γ)						−0.930	0.790	–	−0.020	−0.594	−0.331			
	Incorrectly reports 2 (w_L^ϕ)						−2.881	−3.242	–	−0.000	−0.360	−1.801			
	Expected gain						−1.321	−0.016	–	−0.016	−0.547	−0.625	$D1$	−0.016	
	Correctly reports 3 (w_L^γ)						−0.206	0.309	–	−0.172	0.309	−0.103			
	Incorrectly reports 3 (w_L^ϕ)						0.030	−1.186	–	0.735	−1.186	−2.347			
	Expected gain						−0.159	0.010	–	0.010	0.010	−0.552	$D1$	0.010	0.318

Table 12: One-sided probe with decoys: column player's value

5.2 Two-sided probes with decoys: Single probe

Finally, we allow *both* players to deploy both probes and decoys. We start with the case where both players use only one probe, and we again drop all subscripts associated with μ and ν. We are given probabilities γ and δ of row and column probe successes, respectively, with $\gamma + \delta \le 1$. Probe failure here means that the player sees the opponent's decoy strategy, and $1 - \gamma - \delta$ is the probability that *both* players see a decoy strategy. Neither player, of course, knows whether he or she is looking at the true strategy or the decoy strategy.

The row player's mixed strategy is described by a set $\boldsymbol{p} = (p^{fk})$, $f = 1, \ldots, r$, $k = 1, \ldots, c$, with $p^{fk} \in \mathcal{R}$ and p_i^{fk} representing the probability that the row player will show a strategy in decoy strategy partition f (the precise strategy chosen is irrelevant) and, upon seeing strategy partition k from the column player, will employ true strategy i.

We now calculate the payoff to the row player using strategy $\boldsymbol{p}$ against decoy strategy partition d and response strategy $(j_1, \ldots, j_r)$ of the column player.

Row probe correct: With probability γ the row player's probe is correct, so that his probe indicates the column player's true strategy partition while her probe reports the decoy strategy partition, for which she employs her associated true strategy. The expected payoff for the row player is thus

$$v^{\gamma} = \sum_{f=1}^{r} p^{f\kappa_{j_f}} A^{\mathcal{R},j_f} \tag{10}$$

Column probe correct: With probability δ the column player's probe is correct, so that her probe indicates the row player's true strategy partition while his probe reports the decoy strategy partition. The expected payoff for the row player in this case is

$$v^{\delta} = \sum_{f=1}^{r} \sum_{k=1}^{r} p_{I_k}^{fd} A^{I_k,j_k} \tag{11}$$

Neither probe correct: With probability $1 - \gamma - \delta$ neither player's probe is correct, so that they both see the decoy strategy and play accordingly. The expected payoff for the row player here is

$$v^{\phi} = \sum_{f=1}^{r} p^{fd} A^{\mathcal{R},j_f}. \tag{12}$$

The value of the game for the row player is therefore

$$\gamma v^{\gamma} + \delta v^{\delta} + (1 - \gamma - \delta) v^{\phi}.$$

Now for given row mixed strategy $\boldsymbol{p}$, the column player's best strategy for minimizing this value is to choose the strategy d, $(j_1, \ldots, j_r)$ that minimizes

$$\sum_{f=1}^{r} \gamma p^{f y_f} A^{\mathcal{R},j_f} + \sum_{f=1}^{r} \sum_{k=1}^{r} \delta p_{I_k}^{fd} A^{I_k,j_k} + \sum_{f=1}^{r} (1 - \gamma - \delta) p^{fd} A^{\mathcal{R},j_f} \tag{13}$$

so that the maximin strategy for the row player involves solving the following LP, where ρ_f is the probability that he uses decoy f, v_l^d is part of the game value obtained when the column player uses decoy strategy d and her probe returns partition set l, and $j \in J_y$

represents her true strategy:

$$
\begin{aligned}
&\max v \\
v &\leq \sum_{l=1}^{r} v_l^d,\ d=1,\ldots,c \\
v_l^d &\leq \gamma p^{ly} A^{\mathcal{R},j} + \sum_{f=1}^{r} \delta p_{I_l}^{fd} A^{I_l,j} + (1-\gamma-\delta) p^{ld} A^{\mathcal{R},j}, \\
&\qquad l=1,\ldots,r,\ d=1,\ldots,c,\ y=1,\ldots,c,\ j\in J_y \\
\sum_{i\in\mathcal{R}} p_i^{fk} &= \rho_f,\ f=1,\ldots,r,\quad k=1,\ldots,c \\
p_i^{fk} &\geq 0,\quad \rho_f \geq 0
\end{aligned}
\tag{14}
$$

We can simplify (14) by putting it in symmetric vector form, grouping the I_i terms and $j \in J_l$ constraints:

$$
\begin{aligned}
&\max v \\
v &\leq \sum_{l=1}^{r} v_l^d,\ d=1,\ldots,c \\
v_l^d e_{J_y} &\leq \sum_{x=1}^{r} \gamma p_{I_x}^{ly} A^{I_x J_y} + \sum_{f=1}^{r} \delta p_{I_l}^{fd} A^{I_l J_y} + \sum_{x=1}^{r} (1-\gamma-\delta) p_{I_x}^{ld} A^{I_x J_y}, \\
&= \sum_{x=1}^{r} \left(\gamma p_{I_x}^{ly} A^{I_x J_y} + \delta p_{I_l}^{xd} A^{I_l J_y} + (1-\gamma-\delta) p_{I_x}^{ld} A^{I_x J_l} \right), \\
&\qquad l=1,\ldots,r,\ d=1,\ldots,c,\ y=1,\ldots,c \\
\sum_{x=1}^{r} p_{I_x}^{fk} e^{I_x} &= \rho_f,\ f=1,\ldots,r,\ k=1,\ldots,c \\
p_i^{fk} &\geq 0,\quad \rho_f \geq 0
\end{aligned}
\tag{15}
$$

where e_S (e^S) is the row (column) vector of ones on set S. By attaching dual variable ξ_d to each of the first set of constraints, column vector $q_{J_l}^{ki}$ to each of the second sets of constraints and variable w_k to each of the equality constraints in (15), respectively, we can construct the dual to (15):

$$
\begin{aligned}
&\min w \\
w &\geq \sum_{k=1}^{c} w_k^f,\ f=1\ldots,r \\
e^{I_x} w_k^f &\geq \sum_{y=1}^{c} \left(\delta A^{I_x J_y} q_{J_y}^{xk} + \gamma A^{I_x J_k} q_{J_k}^{fy} + (1-\gamma-\delta) A^{I_x J_y} q_{J_y}^{fk} \right) \\
&= \sum_{y=1}^{c} \gamma A^{I_x J_k} q_{J_k}^{fy} + \sum_{d=1}^{c} \delta A^{I_x J_y} q_{J_y}^{dk} + \sum_{y=1}^{c} (1-\gamma-\delta) A^{I_x J_y} q_{J_y}^{fk} \\
&\qquad k=1,\ldots,c,\ f=1,\ldots,r,\ x=1,\ldots,r \\
\sum_{y=1}^{c} e_{J_y} q_{J_y}^{dl} &= \xi_d,\quad d=1,\ldots,c,\ l=1,\ldots,r \\
q_{J_y}^{dl} &\geq 0,\quad \xi_d \geq 0
\end{aligned}
\tag{16}
$$

This turns out to be exactly the LP for column player's minimax mixed strategy. Thus by solving (16) the column player obtains her optimal strategy, and the two game values are equal.

For our battle example, we take the case where the row player uses the tactic probe and the column player uses the location probe, with probe success probabilities $\gamma = .7$ and $\delta = .1$. Table 13 gives the associated optimal LP solution strategies and game value computations. The organization follows that of Table 12. For the row player's game the expected values are grouped by column probe report as well as column decoy shown. The values computed in the boxes are

v^{γ}: The row strategy decoy group used matches the column report, and the row report choice for that decoy group matches the column strategy.

v^{δ}: The row strategy report column matches the column decoy, and is summed over all of the row decoy groups. The portion of the resulting probability vector used matches the column report partition.

v^{ϕ}: The row strategy decoy group used matches the column report, and the row report choice for that decoy group matches the column decoy.

final row: This represents the average of the above three numbers, weighted by γ, δ, and $1 - \gamma - \delta$, respectively. The minimizing column strategy is chosen for this decoy and report choice, and for each decoy the value for that decoy is determined by summing the corresponding report values.

This shows that the row player can guarantee a loss of no more than .923 over any choice of decoy and response by the column player.

The column player's game tables are symmetric to that of the row player, with w replacing v and the row player playing for maximum gain. The resulting values show that the column player can guarantee a gain of at least .923 as well, and so that two strategies are optimal.

5.3 Two-side probes with decoys: Multiple probes

As in the non-decoy case, we associate with each pair of probes $\mathcal{P}_\mu$ and $\mathcal{Q}_\nu$ probabilities $\gamma_{\mu\nu}$ that the row player's probe is successful and $\delta_{\mu\nu}$ that the column player's probe is successful, with probe failure meaning that the probe reports the opponent's decoy strategy partition instead. Again, $\gamma_{\mu\nu} + \delta_{\mu\nu} \leq 1$ with $1 - \gamma_{\mu\nu} - \delta_{\mu\nu}$ representing the probability that both players probes report the decoy partition. For simplicity we assume that the decoy strategy can use *any* of the player's available strategies, rather than the subset available as a result of employing the probe. Again, both players know their opponents available probes together with all probe-success probabilities, and are interested in maximizing their expected gain over all of their opponents strategies.

The row player's mixed strategy is described by set $\boldsymbol{p}^\mu = (p_j^{\mu f k})$ $\mu = 1, \ldots, s,\ f \in \mathcal{R}$, $k = 1, \ldots, c_\mu$, and $j \in \mathcal{R}_\mu$, where $p_j^{\mu f k}$ represents the probability that he employs probe μ, shows decoy strategy f and, upon seeing strategy partition J_k^μ from the column player, uses true strategy j. Note that, unlike the single-probe case, a player's decoy strategy must be defined on individual strategies, since that player does not know which probe the opponent is using. Again, $\rho_\mu = \sum_{f \in \mathcal{R}} \sum_{l \in \mathcal{R}_\mu} p_l^{\mu f k}$ is the common probability that the row player chooses probe μ.

Now let us consider the effect of using row mixed strategy $\boldsymbol{p}^\mu$, $\mu = 1, \ldots, s$, against the column player's choice of probe ν, decoy strategy d, and response strategies $(j_1, \ldots, j_{r_\nu})$. Following Equations (10), (11), and (12) in the previous section, we get the following expected value for the row player.

	Row Decoy 1		Row Decoy 2		Row Decoy 3		Row Strategies					Final	Report	Decoy
	p^{1A}	p^{1D}	p^{2A}	p^{2D}	p^{3}	p^{3D}	$A1$	$A2$	$D1$	$D2$	$D3$	Strategy	Value (v_l^d)	Value
$A1$	0.000	0.000	0.000	0.000	0.000	0.000	7	−6	−8	−6	−6			
$A3$	0.028	0.009	0.017	0.000	0.185	0.246	−8	3	4	−4	−3			
$D1$	0.031	0.000	0.036	0.035	0.703	0.345	2	−2	0	7	−5			
$D2$	0.000	0.051	0.000	0.018	0.000	0.297	−3	−3	−1	−7	9			
$D3$	0.000	0.000	0.000	0.000	0.000	0.000	−6	−4	−5	5	−3			

			$A1$	$A2$	$D1$	$D2$	$D3$	Final Strategy	Report Value (v_l^d)	Decoy Value
Column Decoy A	Col. Reports 1	row correct (v^γ)	−0.166	0.024	−0.016	−0.389	0.429			
		col. correct (v^δ)	1.540	−1.540	0.000	5.391	−3.850			
		both fail (v^ϕ)	−0.166	0.024	0.114	0.102	−0.240			
		wtd. average	0.005	−0.133	0.012	0.287	−0.133	$D3$	−0.133	
	Col. Reports 2	row correct (v^γ)	−0.061	−0.022	−0.018	0.118	−0.013			
		col. correct (v^δ)	−0.000	−0.000	−0.000	−0.000	0.000			
		both fail (v^ϕ)	−0.061	−0.022	0.066	0.184	−0.229			
		wtd. average	−0.055	−0.020	0.001	0.120	−0.055	$A1$	−0.055	
	Col. Reports 3	row correct (v^γ)	−0.072	−0.852	0.688	−0.652	0.213			
		col. correct (v^δ)	−1.839	0.690	0.920	−0.920	−0.690			
		both fail (v^ϕ)	−0.072	−0.852	0.739	4.185	−4.072			
		wtd. average	−0.249	−0.698	0.721	0.289	−0.735	$D3$	−0.735	−0.923
Column Decoy D	Col. Reports 1	row correct (v^γ)	−0.166	0.024	−0.016	−0.389	0.429			
		col. correct (v^δ)	0.759	−0.759	0.000	2.656	−1.897			
		both fail (v^ϕ)	−0.222	−0.125	−0.016	−0.389	0.429			
		wtd. average	−0.084	−0.084	−0.014	−0.084	0.196	$A1$	−0.084	
	Col. Reports 2	row correct (v^γ)	−0.061	−0.022	−0.018	0.118	−0.013			
		col. correct (v^δ)	−1.097	−1.097	−0.366	−2.559	3.290			
		both fail (v^ϕ)	0.016	−0.123	−0.018	0.118	−0.013			
		wtd. average	−0.149	−0.149	−0.053	−0.149	0.317	$A1$	−0.149	
	Col. Reports 3	row correct (v^γ)	−0.072	−0.852	0.688	−0.652	0.213			
		col. correct (v^δ)	−2.040	0.765	1.020	−1.020	−0.765			
		both fail (v^ϕ)	−2.172	−0.843	0.688	−0.652	0.213			
		wtd. average	−0.689	−0.689	0.721	−0.689	0.115	$D2$	−0.689	−0.923

Row Player's Game

	Column Decoy A			Column Decoy D			Row Strategies					Final	Report	Decoy
	p^{A1}	p^{A2}	p^{A3}	p^{D1}	p^{D2}	p^{D3}	$A1$	$A3$	$D1$	$D2$	$D3$	Strategy	Value (w_k^f)	Value
$A1$	0.000	0.000	0.000	0.309	0.309	0.309	7	−8	2	−3	−6			
$A2$	0.000	0.000	0.000	0.605	0.605	0.605	1	−4	7	1	2			
$D1$	0.000	0.000	0.000	0.000	0.000	0.000	−8	4	0	−1	−5			
$D2$	0.000	0.000	0.000	0.008	0.008	0.008	−6	−4	7	−7	5			
$D3$	0.077	0.077	0.077	0.000	0.000	0.000	−6	−3	−5	9	−3			

			$A1$	$A3$	$D1$	$D2$	$D3$	Final Strategy	Report Value (w_k^f)	Decoy Value
Row Decoy 1	Row Reports A	col. correct (w^δ)	−0.464	−0.232	−0.387	0.696	−0.232			
		row correct (w^γ)	−1.467	−0.658	−0.592	−2.744	−4.277			
		both fail (w^ϕ)	−0.464	−0.232	−0.387	0.696	−0.232			
		wtd. average	−1.166	−0.530	−0.530	−1.712	−3.064	$D1$	−0.530	
	Row Reports D	col. correct (w^δ)	−1.515	−0.691	−0.536	−2.800	−4.237			
		row correct (w^γ)	−0.512	−0.264	−0.331	0.640	−0.192			
		both fail (w^ϕ)	−1.515	−0.691	−0.536	−2.800	−4.237			
		wtd. average	−0.813	−0.392	−0.392	−0.392	−1.405	$A3$	−0.392	−0.923
Row Decoy 2	Row Reports A	col. correct (w^δ)	−0.464	−0.232	−0.387	0.696	−0.232			
		row correct (w^γ)	−1.467	−0.658	−0.592	−2.744	−4.277			
		both fail (w^ϕ)	−0.464	−0.232	−0.387	0.696	−0.232			
		wtd. average	−1.166	−0.530	−0.530	−1.712	−3.064	$D1$	−0.530	
	Row Reports D	col. correct (w^δ)	−1.515	−0.691	−0.536	−2.800	−4.237			
		row correct (w^γ)	−0.512	−0.264	−0.331	0.640	−0.192			
		both fail (w^ϕ)	−1.515	−0.691	−0.536	−2.800	−4.237			
		wtd. average	−0.813	−0.392	−0.392	−0.392	−1.405	$A3$	−0.392	−0.923
Row Decoy 3	Row Reports A	col. correct (w^δ)	−0.464	−0.232	−0.387	0.696	−0.232			
		row correct (w^γ)	−1.467	−0.658	−0.592	−2.744	−4.277			
		both fail (w^ϕ)	−0.464	−0.232	−0.387	0.696	−0.232			
		wtd. average	−1.166	−0.530	−0.530	−1.712	−3.064	$D1$	−0.530	
	Row Reports D	col. correct (w^δ)	−1.515	−0.691	−0.536	−2.800	−4.237			
		row correct (w^γ)	−0.512	−0.264	−0.331	0.640	−0.192			
		both fail (w^ϕ)	−1.515	−0.691	−0.536	−2.800	−4.237			
		wtd. average	−0.813	−0.392	−0.392	−0.392	−1.405	$A3$	−0.392	−0.923

Column Player's Game

Table 13: Two-sided single probe with decoy

Row probe correct: With probability $\gamma_{\mu\nu}$ the row player sees the column player's true strategy, while she sees his decoy strategy, and the expected payoff is

$$v^{\gamma}_{\mu\nu} = \sum_{f \in \mathcal{R}} p^{\mu f \kappa^{\nu}_{j_f}} A^{\mathcal{R}_\mu, j_f}$$

Column probe correct: With probability $\delta_{\mu\nu}$ the row player sees the column player's decoy strategy, while she sees his true strategy, and the expected payoff is

$$v^{\delta}_{\mu\nu} = \sum_{f \in \mathcal{R}} \sum_{k=1}^{r_\nu} p^{\mu f \kappa^{\nu}_{d}}_{I^{\nu\mu}_k} A^{I^{\nu\mu}_k, j_k}$$

Neither probe correct: With probability $1 - \gamma_{\mu\nu} - \delta_{\mu\nu}$ *both* players see decoy strategies, and the expected payoff is

$$v^{\phi}_{\mu\nu} = \sum_{f \in \mathcal{R}} p^{\mu f \kappa^{\nu}_{d}} A^{\mathcal{R}_\mu, j_f}$$

and so the total expected payoff to the row player is

$$\gamma_{\mu\nu} v^{\gamma}_{\mu\nu} + \delta_{\mu\nu} v^{\delta}_{\mu\nu} + (1 - \gamma_{\mu\nu} - \delta_{\mu\nu}) v^{\phi}_{\mu\nu} \; .$$

The column player's best strategy for minimizing his game value against the row player's mixed strategy, then, is to choose these strategies so as to minimize

$$\sum_{\mu=1}^{s} \left(\sum_{f \in \mathcal{R}} \gamma_{\mu\nu} p^{\mu f \kappa^{\nu}_{j_f}} A^{\mathcal{R}_\mu, j_f} + \sum_{f \in \mathcal{R}} \sum_{k=1}^{r_\nu} \delta_{\mu\nu} p^{\mu f \kappa^{\nu}_{d}}_{I^{\nu\mu}_k} A^{I^{\nu\mu}_k, j_k} + \sum_{f \in \mathcal{R}} (1 - \gamma_{\mu\nu} - \delta_{\mu\nu}) p^{\mu f \kappa^{\nu}_{d}} A^{\mathcal{R}_\mu, j_f} \right) .$$

The row player's maximin strategy can now be obtained by solving the following LP, where $\rho_{\mu f}$ is the probability that he chooses probe ν and decoy f, and $v^d_{\nu l}$ is the part of the game value obtained when the column player uses probe ν, decoy d, and sees a strategy in set I^{ν}_l:

$$\begin{aligned}
&\max v \\
&v \;\le\; \sum_{l=1}^{r_\nu} v^d_{\nu l}, \quad \nu = 1, \ldots, t, \; d \in \mathcal{C} \\
&v^d_{\nu l} \;\le\; \sum_{\mu=1}^{s} \left(\sum_{f \in I^{\nu}_l} \gamma_{\mu\nu} p^{\mu f \kappa^{\mu}_{j}} A^{\mathcal{R}_\mu, j} + \sum_{f \in \mathcal{R}} \delta_{\mu\nu} p^{\mu f \kappa^{\mu}_{d}}_{I^{\nu\mu}_l} A^{I^{\nu\mu}_l, j} + \sum_{f \in I^{\nu}_l} (1 - \gamma_{\mu\nu} - \delta_{\mu\nu}) p^{\mu f \kappa^{\mu}_{d}} A^{\mathcal{R}_\mu, j} \right) \\
&\qquad \nu = 1, \ldots, t \text{ (column probe)}, \; l = 1, \ldots, r_\nu \text{ (probe value)}, \\
&\qquad d \in \mathcal{C} \text{ (column decoy)}, \; j \in \mathcal{C}_\nu \text{ (column final strategy)} \\
&\sum_{i \in \mathcal{R}_\mu} p^{\mu f k}_i \;=\; \rho_{\mu f}, \qquad \mu = 1, \ldots, s, \; k = 1, \ldots, c_\mu \; f \in \mathcal{R} \\
&\sum_{\mu=1}^{t} \sum_{f \in \mathcal{C}} \rho_{\mu f} \;=\; 1 \\
&p^{\mu f k}_i \;\ge\; 0
\end{aligned} \tag{17}$$

The dual to (17) is obtained by attaching dual variables $\xi_{\nu d}$, $q^{\nu d l}_j$, $w^f_{\mu k}$, and w to each of the constraints in (17), respectively. With a little work, we get this form for the dual:

$$
\begin{aligned}
&\min w \\
w &\geq \sum_{k=1}^{c_\mu} w^f_{\mu k}, \quad \mu = 1,\ldots,s,\ f \in \mathcal{R} \\
w^f_{\mu k} &\geq \sum_{\nu=1}^{t}\sum_{d\in\mathcal{C}}\sum_{y\in J^\mu_k\cap\mathcal{C}_\nu} \gamma_{\mu\nu} a_{iy} q_y^{\nu d\lambda^\nu_f} + \sum_{\nu=1}^{t}\sum_{d\in J^\mu_k}\sum_{y\in\mathcal{C}_\nu} \delta_{\mu\nu} a_{iy} q_y^{\nu d\lambda^\nu_i} \\
&\qquad\qquad + \sum_{\nu=1}^{t}\sum_{d\in J^\mu_k}\sum_{y\in\mathcal{C}_\nu} (1-\gamma_{\mu\nu}-\delta_{\mu\nu}) a_{xy} q_y^{\nu d\lambda^\nu_f} \\
&= \sum_{\nu=1}^{t}\left(\sum_{d\in J^\mu_k} \delta_{\mu\nu} A^{i,\mathcal{C}_\nu} q^{\nu d\lambda^\nu_i} + \sum_{d\in\mathcal{C}} \gamma_{\mu\nu} A^{i,J^{\mu\nu}_k} q^{\nu d\lambda^\nu_f}_{J^{\mu\nu}_k} + \sum_{d\in J^\nu_k} (1-\gamma_{\mu\nu}-\delta_{\mu\nu}) A^{i,\mathcal{C}_\nu} q^{\nu d\lambda^\nu_f}\right) \\
&\quad \mu = 1,\ldots,s,\ k=1,\ldots,c_\mu,\ f\in\mathcal{R},\ i\in\mathcal{R}_\mu \\
\sum_{j\in\mathcal{C}_\nu} q^{\nu dl}_j &= \xi_{\nu d}, \qquad \nu=1,\ldots,t,\ l=1,\ldots,r_\nu,\ d\in\mathcal{C} \\
\sum_{\nu=1}^{t}\sum_{d\in\mathcal{C}} \xi_{\nu d} &= 1 \\
q^{\nu dl}_j &\geq 0.
\end{aligned}
\tag{18}
$$

Again, (18) turns out to be the symmetric LP for column player's minimax mixed strategy, where

- $q^{\nu dl}_j$ represents probability that she uses probe ν, shows decoy strategy d and, upon seeing column strategy in set I^ν_l, employs true strategy j, and
- $\xi_{\nu d}$ is the probability that she uses probe ν and decoy d.

Thus by solving (18) the column player obtains her optimal strategy, and hence the two game values are equal.

For our battle example, we again use the probe probabilities γ and δ given in Table 7. Table 14 (on the next four pages) gives the associated optimal LP solution strategies and game value computations. These are grouped by row probe, and within each group tables are given for each row decoy. As in Table 11 the expectations are computed for a correct row probe, for both probes failing, and for each correct column probe report. These values are in turn collected by *column* probe and decoy — after multiplying by the appropriate factor $\gamma_{\mu\nu}$, $\delta_{\mu\nu}$, or $1-\gamma_{\mu\nu}-\delta_{\mu\nu}$ — and the weighted sums over all probes and decoys are given in Table 15. From this we get that regardless of the column probe or decoy chosen, the row player can guarantee a loss of no greater than .464.

Table 16 gives the associated probabilities for the column player. We give only the summary column player's values — in Table 17 — to demonstrate that regardless of the row probe or decoy chosen, the column player can guarantee a gain of at least .464, and so both players' strategies are optimal.

Tables for Two-Sided Multiple Probes with Decoys

Row Null Probe Values (Row Decoy D3 Only)

	$p^{N,D3}$	$A1$	$A2$	$A3$	$D1$	$D2$	$D3$
$A1$	0.000	7	−6	1	−8	−6	−6
$A2$	0.026	3	8	−6	−9	−6	9
$A3$	0.000	−8	3	−4	4	−4	−3
$D1$	0.000	2	−2	7	0	7	−5
$D2$	0.000	−3	−3	1	−1	−7	9
$D3$	0.000	−6	−4	2	−5	5	−3
both fail ($v^{\phi}_{00}, v^{\phi}_{0T}, v^{\phi}_{0L}$)		0.077	0.205	−0.154	−0.230	−0.154	0.230
col. tactic correct (v^{δ}_{0T})	A	0.077	−	−0.154	−0.230	−0.154	0.230
	D	0.000	−	0.000	0.000	0.000	0.000
col. location correct (v^{δ}_{0L})	1	0.000	0.000	−	0.000	0.000	0.000
	2	0.077	0.205	−	−0.230	−0.154	0.230
	3	0.000	0.000	−	0.000	0.000	0.000

Row Tactic Probe Values: Row Decoys A1, A2 and A3

Row Decoy A1	$p^{T,A1,A}$	$p^{T,A1,D}$	$A1$	$A2$	$A3$	$D1$	$D2$	$D3$
$A1$	0.000	0.000	7	−6	1	−8	−6	−6
$A3$	0.000	0.029	−8	3	−4	4	−4	−3
$D1$	0.000	0.000	2	−2	7	0	7	−5
$D2$	0.061	0.032	−3	−3	1	−1	−7	9
$D3$	0.000	0.000	−6	−4	2	−5	5	−3
Col. Decoy $A1, A2, A3$	row correct ($v^{\gamma}_{T0}, v^{\gamma}_{TT}, v^{\gamma}_{TL}$)		−0.183	−0.183	0.061	0.085	−0.340	0.198
	both fail ($v^{\phi}_{T0}, v^{\phi}_{TT}, v^{\phi}_{TL}$)		−0.183	−0.183	0.061	−0.061	−0.427	0.550
	col. tactic correct (v^{δ}_{TT})	A	0.000	−	0.000	0.000	0.000	0.000
		D	−0.183	−	0.061	−0.061	−0.427	0.550
	col. location correct (v^{δ}_{TL})	1	0.000	0.000	−	0.000	0.000	0.000
		2	−0.183	−0.183	−	−0.061	−0.427	0.550
		3	0.000	0.000	−	0.000	0.000	0.000
Col. Decoy $D1, D2, D3$	row correct ($v^{\gamma}_{T0}, v^{\gamma}_{TT}, v^{\gamma}_{TL}$)		−0.183	−0.183	0.061	0.085	−0.340	0.198
	both fail ($v^{\phi}_{T0}, v^{\phi}_{TT}, v^{\phi}_{TL}$)		−0.330	−0.008	−0.085	0.085	−0.340	0.198
	col. tactic correct (v^{δ}_{TT})	A	−0.234	−	−0.117	0.117	−0.117	−0.088
		D	−0.095	−	0.032	−0.032	−0.223	0.286
	col. location correct (v^{δ}_{TL})	1	0.000	0.000	−	0.000	0.000	0.000
		2	−0.095	−0.095	−	−0.032	−0.223	0.286
		3	−0.234	0.088	−	0.117	−0.117	−0.088

Row Decoy A2	$p^{T,A2,A}$	$p^{T,A2,D}$	$A1$	$A2$	$A3$	$D1$	$D2$	$D3$
$A1$	0.000	0.000	7	−6	1	−8	−6	−6
$A3$	0.029	0.000	−8	3	−4	4	−4	−3
$D1$	0.382	0.382	2	−2	7	0	7	−5
$D2$	0.000	0.029	−3	−3	1	−1	−7	9
$D3$	0.000	0.000	−6	−4	2	−5	5	−3
Col. Decoy $A1, A2, A3$	row correct ($v^{\gamma}_{T0}, v^{\gamma}_{TT}, v^{\gamma}_{TL}$)		0.530	−0.676	2.557	−0.029	2.469	−1.647
	both fail ($v^{\phi}_{T0}, v^{\phi}_{TT}, v^{\phi}_{TL}$)		0.530	−0.676	2.557	0.117	2.557	−1.998
	col. tactic correct (v^{δ}_{TT})	A	−0.234	−	−0.117	0.117	−0.117	−0.088
		D	0.764	−	2.674	0.000	2.674	−1.910
	col. location correct (v^{δ}_{TL})	1	0.764	−0.764	−	0.000	2.674	−1.910
		2	0.000	0.000	−	0.000	0.000	0.000
		3	−0.234	0.088	−	0.117	−0.117	−0.088
Col. Decoy $D1, D2, D3$	row correct ($v^{\gamma}_{T0}, v^{\gamma}_{TT}, v^{\gamma}_{TL}$)		0.530	−0.676	2.557	−0.029	2.469	−1.647
	both fail ($v^{\phi}_{T0}, v^{\phi}_{TT}, v^{\phi}_{TL}$)		0.676	−0.852	2.703	−0.029	2.469	−1.647
	col. tactic correct (v^{δ}_{TT})	A	0.000	−	0.000	0.000	0.000	0.000
		D	0.676	−	2.703	−0.029	2.469	−1.647
	col. location correct (v^{δ}_{TL})	1	0.764	−0.764	−	0.000	2.674	−1.910
		2	−0.088	−0.088	−	−0.029	−0.205	0.263
		3	0.000	0.000	−	0.000	0.000	0.000

Row Decoy A3	$p^{T,A3,A}$	$p^{T,A3,D}$	$A1$	$A2$	$A3$	$D1$	$D2$	$D3$
$A1$	0.000	0.000	7	−6	1	−8	−6	−6
$A3$	0.000	0.000	−8	3	−4	4	−4	−3
$D1$	0.050	0.050	2	−2	7	0	7	−5
$D2$	0.000	0.000	−3	−3	1	−1	−7	9
$D3$	0.000	0.000	−6	−4	2	−5	5	−3
Col. Decoy $A1, A2, A3$	row correct ($v^{\gamma}_{T0}, v^{\gamma}_{TT}, v^{\gamma}_{TL}$)		0.099	−0.099	0.348	0.000	0.348	−0.248
	both fail ($v^{\phi}_{T0}, v^{\phi}_{TT}, v^{\phi}_{TL}$)		0.099	−0.099	0.348	0.000	0.348	−0.248
	col. tactic correct (v^{δ}_{TT})	A	0.000	−	0.000	0.000	0.000	0.000
		D	0.099	−	0.348	0.000	0.348	−0.248
	col. location correct (v^{δ}_{TL})	1	0.099	−0.099	−	0.000	0.348	−0.248
		2	0.000	0.000	−	0.000	0.000	0.000
		3	0.000	0.000	−	0.000	0.000	0.000
Col. Decoy $D1, D2, D3$	row correct ($v^{\gamma}_{T0}, v^{\gamma}_{TT}, v^{\gamma}_{TL}$)		0.099	−0.099	0.348	0.000	0.348	−0.248
	both fail ($v^{\phi}_{T0}, v^{\phi}_{TT}, v^{\phi}_{TL}$)		0.099	−0.099	0.348	0.000	0.348	−0.248
	col. tactic correct (v^{δ}_{TT})	A	0.000	−	0.000	0.000	0.000	0.000
		D	0.099	−	0.348	0.000	0.348	−0.248
	col. location correct (v^{δ}_{TL})	1	0.099	−0.099	−	0.000	0.348	−0.248
		2	0.000	0.000	−	0.000	0.000	0.000
		3	0.000	0.000	−	0.000	0.000	0.000

Row Tactic Probe Values: Row Decoys D1, D2 and D3

Row Decoy D1	$p^{T,D1,A}$	$p^{T,D1,D}$	$A1$	$A2$	$A3$	$D1$	$D2$	$D3$
$A1$	0.000	0.000	7	−6	1	−8	−6	−6
$A3$	0.046	0.017	−8	3	−4	4	−4	−3
$D1$	0.000	0.000	2	−2	7	0	7	−5
$D2$	0.000	0.029	−3	−3	1	−1	−7	9
$D3$	0.000	0.000	−6	−4	2	−5	5	−3
Col. Decoy $A1, A2, A3$	row correct $(v^{\gamma}_{T0}, v^{\gamma}_{TT}, v^{\gamma}_{TL})$		−0.367	0.138	−0.183	0.037	−0.271	0.214
	both fail $(v^{\phi}_{T0}, v^{\phi}_{TT}, v^{\phi}_{TL})$		−0.367	0.138	−0.183	0.183	−0.183	−0.138
	col. tactic correct (v^{δ}_{TT})	A	−0.367	–	−0.183	0.183	−0.183	−0.138
		D	0.000	–	0.000	0.000	0.000	0.000
	col. location correct (v^{δ}_{TL})	1	0.000	0.000	–	0.000	0.000	0.000
		2	0.000	0.000	–	0.000	0.000	0.000
		3	−0.367	0.138	–	0.183	−0.183	−0.138
Col. Decoy $D1, D2, D3$	row correct $(v^{\gamma}_{T0}, v^{\gamma}_{TT}, v^{\gamma}_{TL})$		−0.367	0.138	−0.183	0.037	−0.271	0.214
	both fail $(v^{\phi}_{T0}, v^{\phi}_{TT}, v^{\phi}_{TL})$		−0.220	−0.038	−0.037	0.037	−0.271	0.214
	col. tactic correct (v^{δ}_{TT})	A	−0.133	–	−0.066	0.066	−0.066	−0.050
		D	−0.088	–	0.029	−0.029	−0.205	0.263
	col. location correct (v^{δ}_{TL})	1	0.000	0.000	–	0.000	0.000	0.000
		2	−0.088	−0.088	–	−0.029	−0.205	0.263
		3	−0.133	0.050	–	0.066	−0.066	−0.050

Row Decoy D2	$p^{T,D2,A}$	$p^{T,D2,D}$	$A1$	$A2$	$A3$	$D1$	$D2$	$D3$
$A1$	0.000	0.000	7	−6	1	−8	−6	−6
$A3$	0.079	0.108	−8	3	−4	4	−4	−3
$D1$	0.000	0.000	2	−2	7	0	7	−5
$D2$	0.029	0.000	−3	−3	1	−1	−7	9
$D3$	0.000	0.000	−6	−4	2	−5	5	−3
Col. Decoy $A1, A2, A3$	row correct $(v^{\gamma}_{T0}, v^{\gamma}_{TT}, v^{\gamma}_{TL})$		−0.717	0.148	−0.285	0.432	−0.432	−0.324
	both fail $(v^{\phi}_{T0}, v^{\phi}_{TT}, v^{\phi}_{TL})$		−0.717	0.148	−0.285	0.285	−0.519	0.027
	col. tactic correct (v^{δ}_{TT})	A	−0.629	–	−0.315	0.315	−0.315	−0.236
		D	−0.088	–	0.029	−0.029	−0.205	0.263
	col. location correct (v^{δ}_{TL})	1	0.000	0.000	–	0.000	0.000	0.000
		2	−0.088	−0.088	–	−0.029	−0.205	0.263
		3	−0.629	0.236	–	0.315	−0.315	−0.236
Col. Decoy $D1, D2, D3$	row correct $(v^{\gamma}_{T0}, v^{\gamma}_{TT}, v^{\gamma}_{TL})$		−0.717	0.148	−0.285	0.432	−0.432	−0.324
	both fail $(v^{\phi}_{T0}, v^{\phi}_{TT}, v^{\phi}_{TL})$		−0.863	0.324	−0.432	0.432	−0.432	−0.324
	col. tactic correct (v^{δ}_{TT})	A	−0.863	–	−0.432	0.432	−0.432	−0.324
		D	0.000	–	0.000	0.000	0.000	0.000
	col. location correct (v^{δ}_{TL})	1	0.000	0.000	–	0.000	0.000	0.000
		2	0.000	0.000	–	0.000	0.000	0.000
		3	−0.863	0.324	–	0.432	−0.432	−0.324

Row Decoy D3	$p^{T,D3,A}$	$p^{T,D3,D}$	$A1$	$A2$	$A3$	$D1$	$D2$	$D3$
$A1$	0.000	0.000	7	−6	1	−8	−6	−6
$A3$	0.000	0.000	−8	3	−4	4	−4	−3
$D1$	0.000	0.000	2	−2	7	0	7	−5
$D2$	0.023	0.023	−3	−3	1	−1	−7	9
$D3$	0.000	0.000	−6	−4	2	−5	5	−3
Col. Decoy $A1, A2, A3$	row correct $(v^{\gamma}_{T0}, v^{\gamma}_{TT}, v^{\gamma}_{TL})$		−0.068	−0.068	0.023	−0.023	−0.159	0.205
	both fail $(v^{\phi}_{T0}, v^{\phi}_{TT}, v^{\phi}_{TL})$		−0.068	−0.068	0.023	−0.023	−0.159	0.205
	col. tactic correct (v^{δ}_{TT})	A	0.000	–	0.000	0.000	0.000	0.000
		D	−0.068	–	0.023	−0.023	−0.159	0.205
	col. location correct (v^{δ}_{TL})	1	0.000	0.000	–	0.000	0.000	0.000
		2	−0.068	−0.068	–	−0.023	−0.159	0.205
		3	0.000	0.000	–	0.000	0.000	0.000
Col. Decoy $D1, D2, D3$	row correct $(v^{\gamma}_{T0}, v^{\gamma}_{TT}, v^{\gamma}_{TL})$		−0.068	−0.068	0.023	−0.023	−0.159	0.205
	both fail $(v^{\phi}_{T0}, v^{\phi}_{TT}, v^{\phi}_{TL})$		−0.068	−0.068	0.023	−0.023	−0.159	0.205
	col. tactic correct (v^{δ}_{TT})	A	0.000	–	0.000	0.000	0.000	0.000
		D	−0.068	–	0.023	−0.023	−0.159	0.205
	col. location correct (v^{δ}_{TL})	1	0.000	0.000	–	0.000	0.000	0.000
		2	−0.068	−0.068	–	−0.023	−0.159	0.205
		3	0.000	0.000	–	0.000	0.000	0.000

Row Location Probe Values: Row Decoys A1, A2 and A3

Row Decoy A1	$p^{L,A1,1}$	$p^{L,A1,2}$	$p^{L,A1,3}$	$A1$	$A2$	$A3$	$D1$	$D2$	$D3$
$A1$	0.000	0.000	0.000	7	−6	1	−8	−6	−6
$A2$	0.000	0.019	0.019	3	8	−6	−9	−6	9
$D1$	0.019	0.000	0.000	2	−2	7	0	7	−5
$D2$	0.000	0.000	0.000	−3	−3	1	−1	−7	9
$D3$	0.000	0.000	0.000	−6	−4	2	−5	5	−3

			$A1$	$A2$	$A3$	$D1$	$D2$	$D3$
Col. Decoy $A1, D1$	row correct $(v^{\gamma}_{L0}, v^{\gamma}_{LT}, v^{\gamma}_{LL})$		0.039	0.155	−0.117	0.000	−0.117	0.175
	both fail $(v^{\phi}_{L0}, v^{\phi}_{LT}, v^{\phi}_{LL})$		0.039	−0.039	0.136	0.000	0.136	−0.097
	col. tactic correct (v^{δ}_{LT})	A	0.000	–	0.000	0.000	0.000	0.000
		D	0.039	–	0.136	0.000	0.136	−0.097
	col. location correct (v^{δ}_{LL})	1	0.039	−0.039	–	0.000	0.136	−0.097
		2	0.000	0.000	–	0.000	0.000	0.000
		3	0.000	0.000	–	0.000	0.000	0.000
Col. Decoy $A2, D2$	row correct $(v^{\gamma}_{L0}, v^{\gamma}_{LT}, v^{\gamma}_{LL})$		0.039	0.155	−0.117	0.000	−0.117	0.175
	both fail $(v^{\phi}_{L0}, v^{\phi}_{LT}, v^{\phi}_{LL})$		0.058	0.155	−0.117	−0.175	−0.117	0.175
	col. tactic correct (v^{δ}_{LT})	A	0.058	–	−0.117	−0.175	−0.117	0.175
		D	0.000	–	0.000	0.000	0.000	0.000
	col. location correct (v^{δ}_{LL})	1	0.000	0.000	–	0.000	0.000	0.000
		2	0.058	0.155	–	−0.175	−0.117	0.175
		3	0.000	0.000	–	0.000	0.000	0.000
Col. Decoy $A3, D3$	row correct $(v^{\gamma}_{L0}, v^{\gamma}_{LT}, v^{\gamma}_{LL})$		0.039	0.155	−0.117	0.000	−0.117	0.175
	both fail $(v^{\phi}_{L0}, v^{\phi}_{LT}, v^{\phi}_{LL})$		0.058	0.155	−0.117	−0.175	−0.117	0.175
	col. tactic correct (v^{δ}_{LT})	A	0.058	–	−0.117	−0.175	−0.117	0.175
		D	0.000	–	0.000	0.000	0.000	0.000
	col. location correct (v^{δ}_{LL})	1	0.000	0.000	–	0.000	0.000	0.000
		2	0.058	0.155	–	−0.175	−0.117	0.175
		3	0.000	0.000	–	0.000	0.000	0.000

Row Decoy A2	$p^{L,A2,1}$	$p^{L,A2,2}$	$p^{L,A2,3}$	$A1$	$A2$	$A3$	$D1$	$D2$	$D3$
$A1$	0.000	0.000	0.000	7	−6	1	−8	−6	−6
$A2$	0.000	0.000	0.000	3	8	−6	−9	−6	9
$D1$	0.000	0.000	0.000	2	−2	7	0	7	−5
$D2$	0.092	0.092	0.092	−3	−3	1	−1	−7	9
$D3$	0.000	0.000	0.000	−6	−4	2	−5	5	−3

			$A1$	$A2$	$A3$	$D1$	$D2$	$D3$
Col. Decoy $A1, D1$	row correct $(v^{\gamma}_{L0}, v^{\gamma}_{LT}, v^{\gamma}_{LL})$		−0.275	−0.275	0.092	−0.092	−0.642	0.826
	both fail $(v^{\phi}_{L0}, v^{\phi}_{LT}, v^{\phi}_{LL})$		−0.275	−0.275	0.092	−0.092	−0.642	0.826
	col. tactic correct (v^{δ}_{LT})	A	0.000	–	0.000	0.000	0.000	0.000
		D	−0.275	–	0.092	−0.092	−0.642	0.826
	col. location correct (v^{δ}_{LL})	1	0.000	0.000	–	0.000	0.000	0.000
		2	−0.275	−0.275	–	−0.092	−0.642	0.826
		3	0.000	0.000	–	0.000	0.000	0.000
Col. Decoy $A2, D2$	row correct $(v^{\gamma}_{L0}, v^{\gamma}_{LT}, v^{\gamma}_{LL})$		−0.275	−0.275	0.092	−0.092	−0.642	0.826
	both fail $(v^{\phi}_{L0}, v^{\phi}_{LT}, v^{\phi}_{LL})$		−0.275	−0.275	0.092	−0.092	−0.642	0.826
	col. tactic correct (v^{δ}_{LT})	A	0.000	–	0.000	0.000	0.000	0.000
		D	−0.275	–	0.092	−0.092	−0.642	0.826
	col. location correct (v^{δ}_{LL})	1	0.000	0.000	–	0.000	0.000	0.000
		2	−0.275	−0.275	–	−0.092	−0.642	0.826
		3	0.000	0.000	–	0.000	0.000	0.000
Col. Decoy $A3, D3$	row correct $(v^{\gamma}_{L0}, v^{\gamma}_{LT}, v^{\gamma}_{LL})$		−0.275	−0.275	0.092	−0.092	−0.642	0.826
	both fail $(v^{\phi}_{L0}, v^{\phi}_{LT}, v^{\phi}_{LL})$		−0.275	−0.275	0.092	−0.092	−0.642	0.826
	col. tactic correct (v^{δ}_{LT})	A	0.000	–	0.000	0.000	0.000	0.000
		D	−0.275	–	0.092	−0.092	−0.642	0.826
	col. location correct (v^{δ}_{LL})	1	0.000	0.000	–	0.000	0.000	0.000
		2	−0.275	−0.275	–	−0.092	−0.642	0.826
		3	0.000	0.000	–	0.000	0.000	0.000

Row Decoy A3	$p^{L,A3,1}$	$p^{L,A3,2}$	$p^{L,A3,3}$	$A1$	$A2$	$A3$	$D1$	$D2$	$D3$
$A1$	0.000	0.000	0.000	7	−6	1	−8	−6	−6
$A2$	0.019	0.000	0.000	3	8	−6	−9	−6	9
$D1$	0.000	0.019	0.019	2	−2	7	0	7	−5
$D2$	0.000	0.000	0.000	−3	−3	1	−1	−7	9
$D3$	0.000	0.000	0.000	−6	−4	2	−5	5	−3

			$A1$	$A2$	$A3$	$D1$	$D2$	$D3$
Col. Decoy $A1, D1$	row correct $(v^{\gamma}_{L0}, v^{\gamma}_{LT}, v^{\gamma}_{LL})$		0.058	−0.039	0.136	−0.175	0.136	−0.097
	both fail $(v^{\phi}_{L0}, v^{\phi}_{LT}, v^{\phi}_{LL})$		0.058	0.155	−0.117	−0.175	−0.117	0.175
	col. tactic correct (v^{δ}_{LT})	A	0.058	–	−0.117	−0.175	−0.117	0.175
		D	0.000	–	0.000	0.000	0.000	0.000
	col. location correct (v^{δ}_{LL})	1	0.000	0.000	–	0.000	0.000	0.000
		2	0.058	0.155	–	−0.175	−0.117	0.175
		3	0.000	0.000	–	0.000	0.000	0.000
Col. Decoy $A2, D2$	row correct $(v^{\gamma}_{L0}, v^{\gamma}_{LT}, v^{\gamma}_{LL})$		0.058	−0.039	0.136	−0.175	0.136	−0.097
	both fail $(v^{\phi}_{L0}, v^{\phi}_{LT}, v^{\phi}_{LL})$		0.039	−0.039	0.136	0.000	0.136	−0.097
	col. tactic correct (v^{δ}_{LT})	A	0.000	–	0.000	0.000	0.000	0.000
		D	0.039	–	0.136	0.000	0.136	−0.097
	col. location correct (v^{δ}_{LL})	1	0.039	−0.039	–	0.000	0.136	−0.097
		2	0.000	0.000	–	0.000	0.000	0.000
		3	0.000	0.000	–	0.000	0.000	0.000
Col. Decoy $A3, D3$	row correct $(v^{\gamma}_{L0}, v^{\gamma}_{LT}, v^{\gamma}_{LL})$		0.058	−0.039	0.136	−0.175	0.136	−0.097
	both fail $(v^{\phi}_{L0}, v^{\phi}_{LT}, v^{\phi}_{LL})$		0.039	−0.039	0.136	0.000	0.136	−0.097
	col. tactic correct (v^{δ}_{LT})	A	0.000	–	0.000	0.000	0.000	0.000
		D	0.039	–	0.136	0.000	0.136	−0.097
	col. location correct (v^{δ}_{LL})	1	0.039	−0.039	–	0.000	0.136	−0.097
		2	0.000	0.000	–	0.000	0.000	0.000
		3	0.000	0.000	–	0.000	0.000	0.000

Row Location Probe Values: Row Decoys D1 and D3

Row Decoy D1	$p^{L,D1,1}$	$p^{L,D1,2}$	$p^{L,D1,3}$	$A1$	$A2$	$A3$	$D1$	$D2$	$D3$
$A1$	0.000	0.000	0.000	7	−6	1	−8	−6	−6
$A2$	0.052	0.033	0.033	3	8	−6	−9	−6	9
$D1$	0.074	0.093	0.093	2	−2	7	0	7	−5
$D2$	0.000	0.000	0.000	−3	−3	1	−1	−7	9
$D3$	0.000	0.000	0.000	−6	−4	2	−5	5	−3
Col. Decoy $A1, D1$	row correct $(v^{\gamma}_{L0}, v^{\gamma}_{LT}, v^{\gamma}_{LL})$			0.304	0.075	0.456	−0.469	0.456	−0.171
	both fail $(v^{\phi}_{L0}, v^{\phi}_{LT}, v^{\phi}_{LL})$			0.304	0.270	0.203	−0.469	0.203	0.101
	col. tactic correct (v^{δ}_{LT})		A	0.156	–	−0.313	−0.469	−0.313	0.469
			D	0.147	–	0.516	0.000	0.516	−0.369
	col. location correct (v^{δ}_{LL})		1	0.147	−0.147	–	0.000	0.516	−0.369
			2	0.156	0.417	–	−0.469	−0.313	0.469
			3	0.000	0.000	–	0.000	0.000	0.000
Col. Decoy $A2, D2$	row correct $(v^{\gamma}_{L0}, v^{\gamma}_{LT}, v^{\gamma}_{LL})$			0.304	0.075	0.456	−0.469	0.456	−0.171
	both fail $(v^{\phi}_{L0}, v^{\phi}_{LT}, v^{\phi}_{LL})$			0.284	0.075	0.456	−0.294	0.456	−0.171
	col. tactic correct (v^{δ}_{LT})		A	0.098	–	−0.196	−0.294	−0.196	0.294
			D	0.186	–	0.652	0.000	0.652	−0.466
	col. location correct (v^{δ}_{LL})		1	0.186	−0.186	–	0.000	0.652	−0.466
			2	0.098	0.262	–	−0.294	−0.196	0.294
			3	0.000	0.000	–	0.000	0.000	0.000
Col. Decoy $A3, D3$	row correct $(v^{\gamma}_{L0}, v^{\gamma}_{LT}, v^{\gamma}_{LL})$			0.304	0.075	0.456	−0.469	0.456	−0.171
	both fail $(v^{\phi}_{L0}, v^{\phi}_{LT}, v^{\phi}_{LL})$			0.284	0.075	0.456	−0.294	0.456	−0.171
	col. tactic correct (v^{δ}_{LT})		A	0.098	–	−0.196	−0.294	−0.196	0.294
			D	0.186	–	0.652	0.000	0.652	−0.466
	col. location correct (v^{δ}_{LL})		1	0.186	−0.186	–	0.000	0.652	−0.466
			2	0.098	0.262	–	−0.294	−0.196	0.294
			3	0.000	0.000	–	0.000	0.000	0.000

(Row Decoy D2 + Location Probe not used)

Row Decoy D3	$p^{L,D3,1}$	$p^{L,D3,2}$	$p^{L,D3,3}$	$A1$	$A2$	$A3$	$D1$	$D2$	$D3$
$A1$	0.000	0.000	0.000	7	−6	1	−8	−6	−6
$A2$	0.000	0.019	0.019	3	8	−6	−9	−6	9
$D1$	0.019	0.000	0.000	2	−2	7	0	7	−5
$D2$	0.000	0.000	0.000	−3	−3	1	−1	−7	9
$D3$	0.000	0.000	0.000	−6	−4	2	−5	5	−3
Col. Decoy $A1, D1$	row correct $(v^{\gamma}_{L0}, v^{\gamma}_{LT}, v^{\gamma}_{LL})$			0.039	0.155	−0.117	0.000	−0.117	0.175
	both fail $(v^{\phi}_{L0}, v^{\phi}_{LT}, v^{\phi}_{LL})$			0.039	−0.039	0.136	0.000	0.136	−0.097
	col. tactic correct (v^{δ}_{LT})		A	0.000	–	0.000	0.000	0.000	0.000
			D	0.039	–	0.136	0.000	0.136	−0.097
	col. location correct (v^{δ}_{LL})		1	0.039	−0.039	–	0.000	0.136	−0.097
			2	0.000	0.000	–	0.000	0.000	0.000
			3	0.000	0.000	–	0.000	0.000	0.000
Col. Decoy $A2, D2$	row correct $(v^{\gamma}_{L0}, v^{\gamma}_{LT}, v^{\gamma}_{LL})$			0.039	0.155	−0.117	0.000	−0.117	0.175
	both fail $(v^{\phi}_{L0}, v^{\phi}_{LT}, v^{\phi}_{LL})$			0.058	0.155	−0.117	−0.175	−0.117	0.175
	col. tactic correct (v^{δ}_{LT})		A	0.058	–	−0.117	−0.175	−0.117	0.175
			D	0.000	–	0.000	0.000	0.000	0.000
	col. location correct (v^{δ}_{LL})		1	0.000	0.000	–	0.000	0.000	0.000
			2	0.058	0.155	–	−0.175	−0.117	0.175
			3	0.000	0.000	–	0.000	0.000	0.000
Col. Decoy $A3, D3$	row correct $(v^{\gamma}_{L0}, v^{\gamma}_{LT}, v^{\gamma}_{LL})$			0.039	0.155	−0.117	0.000	−0.117	0.175
	both fail $(v^{\phi}_{L0}, v^{\phi}_{LT}, v^{\phi}_{LL})$			0.058	0.155	−0.117	−0.175	−0.117	0.175
	col. tactic correct (v^{δ}_{LT})		A	0.058	–	−0.117	−0.175	−0.117	0.175
			D	0.000	–	0.000	0.000	0.000	0.000
	col. location correct (v^{δ}_{LL})		1	0.000	0.000	–	0.000	0.000	0.000
			2	0.058	0.155	–	−0.175	−0.117	0.175
			3	0.000	0.000	–	0.000	0.000	0.000

Table 14: Two-sided multiple probes with decoys: row player's probabilities

Decoy	Expected Gains 1	2	3	4	5	6	Min. Exp. Gain
1	-0.464	-0.464	2.817	-0.464	1.178	-0.464	-0.464
2	-0.464	-0.464	2.817	-0.464	1.178	-0.464	-0.464
3	-0.464	-0.464	2.817	-0.464	1.178	-0.464	-0.464
4	-0.464	-0.464	2.817	-0.464	1.178	-0.464	-0.464
5	-0.464	-0.464	2.817	-0.464	1.178	-0.464	-0.464
6	-0.464	-0.464	2.817	-0.464	1.178	-0.464	-0.464

Column Null Probe

Decoy	Report	Expected Gains 1	3	4	5	6	Minimum Exp. Gain	Exp. Gain for Decoy
1	A	-0.239	1.390	-0.239	0.583	-0.239	-0.239	-0.464
	D	-0.226	1.426	-0.226	0.595	-0.226	-0.226	
2	A	-0.239	1.390	-0.239	0.583	-0.239	-0.239	-0.464
	D	-0.226	1.426	-0.226	0.595	-0.226	-0.226	
3	A	-0.239	1.390	-0.239	0.583	-0.239	-0.239	-0.464
	D	-0.226	1.426	-0.226	0.595	-0.226	-0.226	
4	A	-0.239	1.390	-0.239	0.583	-0.239	-0.239	-0.464
	D	-0.226	1.426	-0.226	0.595	-0.226	-0.226	
5	A	-0.239	1.390	-0.239	0.583	-0.239	-0.239	-0.464
	D	-0.226	1.426	-0.226	0.595	-0.226	-0.226	
6	A	-0.239	1.390	-0.239	0.583	-0.239	-0.239	-0.464
	D	-0.226	1.426	-0.226	0.595	-0.226	-0.226	

Column Tactic Probe

Decoy	Report	Expected Gains 1	2	4	5	6	Minimum Exp. Gain	Exp. Gain for Decoy
1	1	-0.125	-0.125	-0.125	0.316	-0.125	-0.125	-0.464
	2	-0.293	-0.293	-0.293	0.744	-0.293	-0.293	
	3	-0.046	-0.046	-0.046	0.118	-0.046	-0.046	
2	1	-0.125	-0.125	-0.125	0.316	-0.125	-0.125	-0.464
	2	-0.293	-0.293	-0.293	0.744	-0.293	-0.293	
	3	-0.046	-0.046	-0.046	0.118	-0.046	-0.046	
3	1	-0.125	-0.125	-0.125	0.316	-0.125	-0.125	-0.464
	2	-0.293	-0.293	-0.293	0.744	-0.293	-0.293	
	3	-0.046	-0.046	-0.046	0.118	-0.046	-0.046	
4	1	-0.125	-0.125	-0.125	0.316	-0.125	-0.125	-0.464
	2	-0.293	-0.293	-0.293	0.744	-0.293	-0.293	
	3	-0.046	-0.046	-0.046	0.118	-0.046	-0.046	
5	1	-0.125	-0.125	-0.125	0.316	-0.125	-0.125	-0.464
	2	-0.293	-0.293	-0.293	0.744	-0.293	-0.293	
	3	-0.046	-0.046	-0.046	0.118	-0.046	-0.046	
6	1	-0.125	-0.125	-0.125	0.316	-0.125	-0.125	-0.464
	2	-0.293	-0.293	-0.293	0.744	-0.293	-0.293	
	3	-0.046	-0.046	-0.046	0.118	-0.046	-0.046	

Column Location Probe

Table 15: Two-sided multiple probes with decoys: row player's summary tables

Null Probe: not employed

Tactic Probe: only employed with Decoy $D3$

$$p_{A1}^{T,D3,A} = p_{A1}^{T,D3,D} = .273 \text{ is the only strategy used}$$

Location Probe:

	Row Decoy $A1$			Row Decoy $A2$			Row Decoy $A3$		
	$q^{L,A1,1}$	$q^{L,A1,2}$	$q^{L,A1,3}$	$q^{L,A2,1}$	$q^{L,A2,2}$	$q^{L,A2,3}$	$q^{L,A3,1}$	$q^{L,A3,2}$	$q^{L,A3,3}$
$A1$	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.002
$A2$	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.033
$D1$	0.000	0.000	0.000	0.145	0.145	0.144	0.035	0.035	0.000
$D2$	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
$D3$	0.123	0.123	0.123	0.017	0.017	0.018	0.000	0.000	0.000

	Row Decoy $D1$			Row Decoy $D2$			Row Decoy $D3$		
	$q^{L,D1,1}$	$q^{L,D1,2}$	$q^{L,D1,3}$	$q^{L,D2,1}$	$q^{L,D2,2}$	$q^{L,D2,3}$	$q^{L,D3,1}$	$q^{L,D3,2}$	$q^{L,D3,3}$
$A1$	0.000	0.000	0.000	0.000	0.000	0.000	0.004	0.004	0.002
$A2$	0.056	0.056	0.056	0.000	0.000	0.000	0.102	0.102	0.069
$D1$	0.000	0.000	0.000	0.000	0.000	0.001	0.245	0.245	0.280
$D2$	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
$D3$	0.000	0.000	0.000	0.001	0.001	0.000	0.000	0.000	0.000

Table 16: Two-sided multiple probes with decoys: column player's probabilities

Decoy	Expected Gains 1	2	3	4	5	6	Min. Exp. Gain
1	-3.250	-0.464	-0.464	-0.464	-0.464	-4.838	-0.464
2	-3.250	-0.464	-0.464	-0.464	-0.464	-4.838	-0.464
3	-3.250	-0.464	-0.464	-0.464	-0.464	-4.838	-0.464
4	-3.250	-0.464	-0.464	-0.464	-0.464	-4.838	-0.464
5	-3.250	-0.464	-0.464	-0.464	-0.464	-4.838	-0.464
6	-3.250	-0.464	-0.464	-0.464	-0.464	-4.838	-0.464

Column Null Probe

Decoy	Report	Expected Gains 1	3	4	5	6	Minimum Exp. Gain	Exp. Gain for Decoy
1	A	-0.755	-0.262	-0.262	-0.262	-1.343	-0.262	-0.464
	D	-2.495	-0.203	-0.203	-0.203	-3.496	-0.203	
2	A	-0.755	-0.262	-0.262	-0.262	-1.343	-0.262	-0.464
	D	-2.495	-0.203	-0.203	-0.203	-3.496	-0.203	
3	A	-0.734	-0.275	-0.275	-0.275	-1.336	-0.275	-0.464
	D	-2.515	-0.190	-0.190	-0.190	-3.502	-0.190	
4	A	-0.755	-0.262	-0.262	-0.262	-1.343	-0.262	-0.464
	D	-2.495	-0.203	-0.203	-0.203	-3.496	-0.203	
5	A	-0.755	-0.262	-0.262	-0.262	-1.343	-0.262	-0.464
	D	-2.495	-0.203	-0.203	-0.203	-3.496	-0.203	
6	A	-0.734	-0.275	-0.275	-0.275	-1.336	-0.275	-0.464
	D	-2.515	-0.190	-0.190	-0.190	-3.502	-0.190	

Column Tactic Probe

Decoy	Report	Expected Gains 1	2	4	5	6	Minimum Exp. Gain	Exp. Gain for Decoy
1	1	-0.843	-0.104	-0.104	-0.104	-2.196	-0.104	-0.464
	2	-1.139	-0.180	-0.180	-0.180	-0.720	-0.180	
	3	-1.267	-0.180	-0.180	-0.180	-1.922	-0.180	
2	1	-0.843	-0.104	-0.104	-0.104	-2.196	-0.104	-0.464
	2	-1.139	-0.180	-0.180	-0.180	-0.720	-0.180	
	3	-1.267	-0.180	-0.180	-0.180	-1.922	-0.180	
3	1	-0.843	-0.104	-0.104	-0.104	-2.196	-0.104	-0.464
	2	-1.139	-0.180	-0.180	-0.180	-0.720	-0.180	
	3	-1.267	-0.180	-0.180	-0.180	-1.922	-0.180	
4	1	-0.843	-0.104	-0.104	-0.104	-2.196	-0.104	-0.464
	2	-1.139	-0.180	-0.180	-0.180	-0.720	-0.180	
	3	-1.267	-0.180	-0.180	-0.180	-1.922	-0.180	
5	1	-0.843	-0.104	-0.104	-0.104	-2.196	-0.104	-0.464
	2	-1.139	-0.180	-0.180	-0.180	-0.720	-0.180	
	3	-1.267	-0.180	-0.180	-0.180	-1.922	-0.180	
6	1	-0.843	-0.104	-0.104	-0.104	-2.196	-0.104	-0.464
	2	-1.139	-0.180	-0.180	-0.180	-0.720	-0.180	
	3	-1.267	-0.180	-0.180	-0.180	-1.922	-0.180	

Column Location Probe

Table 17: Two-sided multiple probes with decoys: column player's summary tables

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