A PRODUCTION-INVENTORY SYSTEM IN A STOCHASTIC ENVIRONMENT WITH OR WITHOUT AN EXTERNAL SUPPLIER

Vidyadhar G. Kulkarni [*], Nan Liu

Department of Statistics and Operations Research,

University of North Carolina at Chapel Hill,

Chapel Hill, NC, 27599.

Email: vkulkarn@email.unc.edu, nliu@email.unc.edu.

Short title: A Production-Inventory System in A Stochastic Environment

[*]Corresponding author

Abstract

We consider a production-inventory system where the production and demand follow Poisson processes whose rates are modulated by an external stochastic environment process. We study two such systems: with and without an external supplier, and show an interesting stochastic decomposition relationship between the steady state distribution of inventory levels in these two systems. We consider the optimization problem of the system with an external supplier, and provide a closed form formula similar to the classical EOQ one to compute the optimal order size. Finally we illustrate our results through an application to an integrated warranty inventory management problem.

1 Introduction

In this paper we study a production-inventory system where the production and demand are modulated by an external environment process, e.g. economic situation, sales season, weather etc. In particular, we model the production and demand as Poisson processes whose rates determined by the environment process. We model the environment process as a Continuous Time Markov Chain (CTMC) with finite states. A production or demand can trigger the environment process to jump to another sate instantaneously (or stay unchanged) with a given probability. We study two cases. The first one is corresponding to a lost-sale inventory system without an external supplier where the inventory level remains 0 if its current level is 0 and a demand occurs. The second one corresponds to an inventory system with an external supplier where the inventory is replenished to a random level immediately after a demand occurs when the inventory level is 0. This system behaves as an inventory system under continuous review where no backlogging is allowed, leadtimes are zero, an order is placed when the inventory level hits -1 and the yield is uncertain (the replenishment level is random).

Our primary motivation to study these models is to provide theoretical insights to production-inventory systems modulated by a Morkovian environment. We study the steady state distribution of the inventory level for both models and show an interesting decomposition relationship between them. Given certain cost measures and dependence structures of the yield (replenishment level) and the order size, we study the optimal size of the order from the external supplier when a demand occurs at inventory level 0.

The inventory control in a stochastic environment has been studied extensively in literatures. Scarf [9] and Zipkin [12] provide a detailed review on the inventory problem under continuous review with discrete production and demand. Song and Zipkin [11] consider the inventory control problem under Markov modulated Poisson demands and derive the structure of the optimal ordering policy to be an environment-dependent one. A recent paper by Schwarz et al. [10] studies an inventory control problem where the demand is modulated by an $M/M/1$ queue. In the case of continuous fluid model, Browne and Zipkin [2] study a model with continuous demand driven by a Markov process. Fluid version of these models are studied in Kulkarni, Tzenova and Adan [5] and Kulkarni and Yan [6] and [7]. The closest paper to our analysis is [7] where the authors consider a continuous fluid-flow system with jumps at the boundary. However, to our best knowledge, none of literatures before have shown the stochastic decomposition property in the systems we study here or investigated the optimal order size as in this paper.

Furthermore, an interesting warranty inventory management model is presented in the paper as a direct application of the theoretical results. Warranty has been studied to a great extent in past and related literatures are scattered across many journals from different disciplines. Blischke and Murthy provide a detailed review in [1]. In our model, the failed product under warranty receives a Free Replacement, which generates a demand to the inventory. Hence the demands are modulated by the products under warranty, which are modeled as a multi-variate CTMC in our setup. We also consider the case where the demand is a combined one, which consists of both sales and replacement requests.

The remainder of the paper is organized as follows. In section 2 we introduce the preliminary results on the system without an external supplier. In section 3, we study the system with an external supplier and show the stochastic decomposition property. The optimization of the system with an external supplier is presented in section 4, which is followed by the case study of warranty inventory model in section 5. Some possible extensions are discussed in section 6.

2 The Model without Jumps

In this section we consider a production-inventory system in a stochastic environment, and without an external supplier. We call it the model without jumps. Let $Y(t)$ be the state of the environment at time t. We assume that $\{Y(t), t \geq 0\}$ is a stochastic process on a finite state space $\Omega = \{1, 2, ...n\}$. During the time intervals when $Y(t) = i$, the production occurs according to $PP(\lambda_i)$ and the demands occur according to $PP(\mu_i)$, where $PP(\gamma)$ represents a Poisson process with rate parameter γ. Let $S_0 = 0$ and S_k be the kth production or demand event. We assume that over (S_k, S_{k+1}) the process $\{Y(t), t \in (S_k, S_{k+1})\}$ behaves as an irreducible Continuous Time Markov Chain (CTMC) with generator matrix $Q = [q_{i,j}]$. Furthermore, when a production (or a demand) occurs at time S_k, the $\{Y(t), t \geq 0\}$ process changes instantaneously with transition probabilities given by:

$$a_{i,j} = Pr[Y(S_k^+) = j | Y(S_k^-) = i, \text{ the } k\text{th event is a production}], \tag{2.1}$$

$$b_{i,j} = Pr[Y(S_k^+) = j | Y(S_k^-) = i, \text{ the } k\text{th event is a demand}]. \tag{2.2}$$

It is not difficult to see that $\{Y(t), t \geq 0\}$ is a CTMC on Ω. We denote that $A = [a_{i,j}]$ and $B = [b_{i,j}]$ are $n \times n$ matrices, $\gamma = [\gamma_i]$ are $1 \times n$ row vector whose ith entry is γ_i, and for a row vector γ, $\Delta(\gamma)$ is a diagonal matrix whose (i,i)th entry is γ_i. With these notations, the generator matrix of $\{Y(t), t \geq 0\}$ is given by:

$$Q_Y = Q - \Delta(\lambda + \mu) + \Delta(\lambda) \cdot A + \Delta(\mu) \cdot B. \tag{2.3}$$

Note that in general $Q_Y \neq Q$ unless $A = B = I$ where I is an identity matrix. Since Q is the generator matrix of an irreducible CTMC, so is Q_Y. Let $\pi = [\pi_i]$ be the unique solution to:

$$\begin{aligned} \pi Q_Y &= 0, \\ \pi e &= 1, \end{aligned} \tag{2.4}$$

where 0 is a row vector with all 0's and e is a collum vector with all 1's.

Let $X(t)$ be the inventory level at time t. In this section we assume that the unsatisfied demand is lost so $\{X(t), t \geq 0\}$ has state space $S = \{0, 1, 2...\}$. It is easy to see that $\{(X(t), Y(t)), t \geq 0\}$ is a bi-variate CTMC on the state space $S \times \Omega$. We assume that it is irreducible. Let $r_i = \lambda_i - \mu_i$ be the net demand rate at state i, $\forall i$, and $r = [r_i]$. By theorem 3.1.1 in Neuts [8], the system is stable if and only if the expected net production rate in steady state is negative, i.e.

$$\pi r^T < 0. \tag{2.5}$$

Assume the stability condition (2.5) holds. Let

$$p_{i,j} = \lim_{t \to \infty} Pr[(X(t), Y(t)) = (i, j)], \quad (i, j) \in S \times \Omega. \tag{2.6}$$

Then $[p_{i,j}]$ is the unique solution to the following linear system:

$$
\begin{aligned}
(\sum_{k\neq j} q_{j,k} + \lambda_j + \mu_j)p_{0,j} &= \sum_{k\neq j} p_{0,k}q_{k,j} + \sum_k \mu_k b_{k,j} p_{1,k} \\
&\quad + \sum_k \mu_k b_{k,j} p_{0,k} \qquad \forall j,
\end{aligned} \tag{2.7}
$$

$$
\begin{aligned}
(\sum_{k\neq j} q_{j,k} + \lambda_j + \mu_j)p_{i,j} &= \sum_{k\neq j} p_{i,k}q_{k,j} + \sum_k \lambda_k a_{k,j} p_{i-1,k} \\
&\quad + \sum_k \mu_k b_{k,j} p_{i+1,k} \quad \forall i \geq 1, \forall j,
\end{aligned} \tag{2.8}
$$

$$
\sum_{i,j} p_{i,j} = 1. \tag{2.9}
$$

One can compute the limiting distribution $[p_{i,j}]$ through the general procedures discussed in section 1.9 of Neuts [8]. Here we consider the generating function method. Let $\psi^j(z) = \sum_i z^i p_{i,j}$ and $\psi(z) = [\psi^1(z), \psi^2(z), ..., \psi^n(z)]$. Denote that $p_{0,\cdot} = [p_{0,1}, p_{0,2}, ..., p_{0,n}]$, $\nu = p_{0,\cdot}[\Delta(\mu)\cdot B]$ and $U(z) = \Delta(\lambda+\mu) - z[\Delta(\lambda)\cdot A] - \frac{1}{z}[\Delta(\mu)\cdot B]$. Following the standard steps, from Eq. (2.7) and (2.8) we obtain:

$$
\psi(z) = \frac{z-1}{z}\nu[U(z) - Q]^{-1}. \tag{2.10}
$$

Notice that:

$$
[U(z) - Q]^{-1} = \frac{adj(U(z) - Q)}{det(U(z) - Q)}, \tag{2.11}
$$

where $adj(A)$ represents the adjoint of a matrix A and $det(A)$ denotes the determinant of a matrix A.

By *Rouché's theorem* in complex analysis, one can show that $det(U(z) - Q)$ has exactly n roots $\{z_1, z_2, ..., z_n\}$ in the unit disk with $z_1 = 1$ and $|z_k| < 1$ for $k = 2, ..., n$. See Eq.

(1.6.5) of Neuts [8] for more details. Since the probability generation function vector $\psi(z)$ is analytic for $|z| \leq 1$, the numerator in Eq. (2.10) must be zero for $z = z_k$, $k = 2, ..., n$. So we have:

$$\nu[adj(U(z_k) - Q)] = 0, k = 2, ..., n. \tag{2.12}$$

We assume that for $k = 2, ..., n$, z_k are distinct from each other. Thus $U(z_k) - Q$ for $k = 2, 3, ..., n$ has rank $n - 1$. Notice that $adj(U(z) - Q)$ has rank one if $U(z) - Q$ has rank $n - 1$. Hence from the linear system (2.12) we obtain $n - 1$ equations. With the normalizing condition $\lim_{z \to 1} \sum_{j=1}^{n} \psi^j(z) = 1$, we can compute the values of $p_{0,\cdot}$. Since $[p_{i,j}]$ is the unique solution to the system (2.7), (2.8) and (2.9), it follows that there exists a unique vector ν which makes $\psi(z)$ a valid generating function vector of random variables. We shall use this fact in deriving results in section 3.

Remark 2.1. Neuts [8] presents a similar problem in chapter 6 where the state of the environment does not change when a production or demand occurs.

3 The Model with Jumps

In this section, we consider a production-inventory system in a stochastic environment, and with an external supplier. We call it the model with jumps. To distinguish from the model without jumps, let $X'(t)$ be the inventory level at time t and $Y'(t)$ denote the state of the environment at time t. This model behaves almost the same as the model without jumps except for an interesting "jumping" behavior which is described as follows. When a demand occurs at time t and the inventory level $X'(t^-) = 0$, $X'(t)$ will first backlog this demand and decrease to level -1, then jump up by J immediately, so $X'(t^+) = J - 1$, where J is a discrete random variable defined on $\mathbb{Z}^+ = \{1, 2, 3...\}$. Essentially in this situation, $X'(t)$ spends no time in level -1 but jumps from 0 to $J - 1$ immediately. We may think of this "jumping" behavior as an order placement, i.e. whenever the inventory level hits level -1 we place an order, but the yield is uncertain so we actually receive a random amount J of units and the inventory level is replenished to $J - 1$. As we can see, this corresponds to an inventory model without backlogging nor leadtime.

It is clear that $\{(X'(t), Y'(t)), t \geq 0\}$ is a bi-variate CTMC on state space $S \times \Omega$. We assume that it is irreducible. The next theorem states the stability condition for the $\{(X'(t), Y'(t)), t \geq 0\}$ process.

Theorem 3.1. *The* $\{(X'(t), Y'(t)), t \geq 0\}$ *process is stable if:*

$$\begin{aligned} \pi r^T &< 0, \\ Pr[J < \infty] &= 1. \end{aligned} \tag{3.1}$$

Proof:

Let $S_0 = 0$ and S_k represent the kth jump of the $X'(t)$ process from 0 to $J-1$, i.e. $X(S_k^-) = 0$ and $X(S_k^+) = J-1$. Let $Z_k = \{X'(S_k^+), Y'(S_k^+)\}$. It is easy to see that $\{(Z_k, S_k), k \geq 0\}$ is a Markov renewal sequence and $\{(X'(t), Y'(t)), t \geq 0\}$ is the corresponding Markov regenerative process (MRGP). From the theory of MRGP, the limiting distribution of

$\{(X'(t), Y'(t)), t \geq 0\}$ exists if $E(S_k) < \infty$. By the stability condition (2.5) as discussed in section 2, we know that $E(S_k) < \infty$ if $\pi r^T < 0$ and $Pr[J < \infty] = 1$. This proves the theorem. ■.

3.1 Stochastic Decomposition Property

Let $X' \stackrel{d}{=} \lim_{t\to\infty} X'(t)$ and $X \stackrel{d}{=} \lim_{t\to\infty} X(t)$, where $\stackrel{d}{=}$ represents equality in distribution. The next theorem gives an interesting stochastic decomposition property between X' and X.

Theorem 3.2. *Assume the stability conditions (3.1) hold, then*

$$X' \stackrel{d}{=} X_0 + X, \tag{3.2}$$

where:

1. $Pr(X_0 = i) = \frac{Pr(J>i)}{E(J)}$, $i \in \mathbb{Z}^+ \cup \{0\}$,

2. X_0 *and* X *are independent.*

Proof:

Let $\alpha_i = Pr(J = i), i \in \mathbb{Z}^+$ and $p'_{i,j} = \lim_{t\to\infty} Pr[(X'(t), Y'(t)) = (i, j)] \quad (i, j) \in S \times \Omega$. We have the following balance equations corresponding to the system with an external supplier:

$$
\begin{aligned}
(\sum_{k\neq j} q_{j,k} + \lambda_j + \mu_j) p'_{0,j} &= \sum_{k\neq j} p'_{0,k} q_{k,j} + \alpha_1 \sum_k \mu_k b_{k,j} p'_{0,k} \qquad (3.3)\\
&\quad + \sum_k \mu_k b_{k,j} p'_{1,k} \qquad \forall j,\\
(\sum_{k\neq j} q_{j,k} + \lambda_j + \mu_j) p'_{i,j} &= \sum_{k\neq j} p'_{i,k} q_{k,j} + \alpha_{(i+1)} \sum_k \mu_k b_{k,j} p'_{0,k} \qquad (3.4)\\
&\quad + \sum_k \lambda_k a_{k,j} p'_{i-1,k} + \sum_k \mu_k b_{k,j} p'_{i+1,k} \quad \forall i \geq 1, \forall j.
\end{aligned}
$$

Denote that $\omega = p'_{0,\cdot}[\Delta(\mu) \cdot B]$. Let $\phi^j(z) = \sum_i z^i p'_{i,j}$, $\phi(z) = [\phi^1(z), \phi^2(z), ..., \phi^n(z)]$ and $\phi_J(z) = E(z^J)$. Through some elementary but tedious algebra, we obtain the follows from Eq. (3.3) and (3.4):

$$
\phi(z) = \frac{\phi_J(z) - 1}{z} \omega [U(z) - Q]^{-1}. \qquad (3.5)
$$

Rewrite Eq. (3.5) into the following form:

$$
\phi(z) = \frac{1}{E(J)} \frac{\phi_J(z) - 1}{z - 1} E(J) \frac{z - 1}{z} \omega [U(z) - Q]^{-1}. \qquad (3.6)
$$

Notice that $\phi(z)$ is a valid generating function of random vector and $\frac{1}{E(J)} \frac{\phi_J(z)-1}{z-1}$ is the generating function of random variable X_0, where $Pr(X_0 = i) = \frac{Pr(J>i)}{E(J)}$, $i \in \mathbb{Z}^+ \cup \{0\}$. Then $E(J)\frac{z-1}{z}\omega[U(z)-Q]^{-1}$ must constitute a valid generating function of random vector. Recall the form of Eq. (2.10). Since ν is the unique vector which makes $\psi(z)$ of Eq. (2.10) a valid generating function of random vector, we must have $\nu = E(J)\omega$. Hence the results follow. ■.

Remark 3.1. Theorem 3.2 implies that in steady state, the inventory level in the model with jumps is the sum of two independent random variables: one only depends on J and the other is the steady state inventory level in the model without jumps.

Remark 3.2. If J is deterministic, i.e. $Pr(J = q) = 1$ for some $q \geq 1$, then it is easy to verify that X_0 is a discrete uniform random variable on $\{0, 1, ..., q-1\}$.

Remark 3.3. Consider a special case where $\max_i \lambda_i = 0$ and $\max_i \mu_i > 0$, then the system without an external supplier (as discussed in section 2) is stable but reducible, and $p_{0,j} = \pi_j \ \forall j$ and $p_{i,j} = 0$, $\forall i \geq 1, \forall j$. It can be shown that $[p_{i,j}]$ is the unique solution to the linear system (2.7), (2.8) and (2.9). So theorem 3.2 still holds for this special case. Notice that in this situation, $X = 0$ with probability 1 so $X' \stackrel{d}{=} X_0$. Since X_0 depends on J only and is independent with the $\{Y'(t), t \geq 0\}$ process, so the limiting distribution of inventory level is independent with the external environment process. This is a quite useful result, which simplifies the computation of average holding cost for an inventory system where the demand process is modulated by a CTMC. We will use this result in section 4.

Remark 3.4. It can be easily shown that the stochastic decomposition property also holds in a backlogging inventory system. Suppose the reorder point is r rather than -1, then we have a general backlogging system. To follow the same proof and show this property, we only need to re-index the $\{X'(t), t \geq 0\}$ process, i.e. state 0 in the non-backlogging system is renumbered as state $r + 1$ in the backlogging system, and so on and so forth.

Remark 3.5. Lemma 6.3.1 in Zipkin [12, p. 201] shows that if one manages an inventory system in a Markovian demand environment using a (r, q) policy, then the inventory level is uniformly distributed on $\{r+1, r+2, ..., r+q\}$ in limiting state. J is deterministic and there is no production in that model, hence it is a special case of theorem 3.2, remarks 3.2, 3.3 and 3.4.

Remark 3.6. It can be shown that if the demand and production are compound Poisson processes rather than Poisson processes, the stochastic decomposition property still holds.

Remark 3.7. A sample path argument similar to Kulkarni and Yan [7] may also be used to prove this theorem.

3.2 Average Cost Computation

Recall the model with jumps: the inventory will jump from 0 to $J-1$ when a demand occurs at time t and $X'(t^-) = 0$. This system behaves as if one manages the inventory through a continuous review and using a (r, q) policy where $r = -1$. However, this operates in a different fashion from the traditional (r, q) policy due to the uncertain yield, i.e. we place a "nominal" order size q but only receive an "effective" order size J. To capture this dependence relationship, we explicitly write the "effective" order size J given a "nominal" order size q as $J|q$. For convenience, we call this type of policy as q-policy. In this section we show how to compute the long run average cost of following a q-policy and in section

4 we present how to find the optimal q-policy which minimizes the long run average cost.

First we introduce the cost structure. Assume that we have no control on the production which is an exogenous process, so we only need to consider the ordering and holding cost. We use the following notations. The fixed ordering cost is K. The procurement cost per unit is c and holding cost rate per unit is h.

Suppose we order q units, we will actually receive a random quantity $J|q$ of units. Let the total cost rate $c(J|q) = h(J|q)+k(J|q)+p(J|q)$, where $h(J|q)$, $k(J|q)$ and $p(J|q)$ represent the holding cost, ordering cost and procurement cost per unit time in steady state.

First we compute $h(J|q)$. From Theorem 3.2, we have:

$$h(J|q) = E(X')h = E(X_0)h + E(X)h = (\frac{E(J^2|q)}{2E(J|q)} - \frac{1}{2})h + E(X)h. \tag{3.7}$$

Note that the second term $E(X)h$ is a constant independent of q, which plays no role in the optimization. Next we compute $k(J|q)$. Let T be the time interval between two consecutive order placements in steady state. From the results on renewal reward processes we get:

$$k(J|q) = \frac{K}{E(T)}. \tag{3.8}$$

In steady state the average net demand during T should be equal to the external supply. Let Δ denote the average net demand rate in steady state. It is given by:

$$\Delta = -\pi r^T. \tag{3.9}$$

Then we have:

$$E(T)\Delta = E(J|q). \tag{3.10}$$

Thus:

$$k(J|q) = \frac{K\Delta}{E(J|q)}. \tag{3.11}$$

Similar to Eq. (3.8), we have:

$$p(J|q) = \frac{cE(J|q)}{E(T)} = c\Delta. \tag{3.12}$$

Note that $p(J|q)$ is also independent of q. Hence,

$$c(J|q) = \frac{E(J^2|q)}{2E(J|q)}h + \frac{K\Delta}{E(J|q)} + \Lambda, \tag{3.13}$$

where $\Lambda = (E(X) - \frac{1}{2})h - c\pi r^T$ is independent of q. Instead of optimizing $c(J|q)$, we could simply optimize $\tilde{c}(J|q) = c(J|q) - \Lambda$.

For convenience, rewrite $\tilde{c}(J|q)$ in the following form:

$$\tilde{c}(J|q) = \frac{E(J|q)}{2}h + \frac{\frac{1}{2}Var(J|q)h + K\Delta}{E(J|q)}. \tag{3.14}$$

4 Optimization of the Model with Jumps

In this section we study the problem of finding an optimal q-policy. We consider two cases. In both cases we assume immediate reimbursement on defective units, i.e. regardless of the "nominal" order size q we only pay for the "effective" order size $J|q$. In the first case we assume that we know the distribution of $J|q$ and need to find the optimal q denoted as q^*. This case mimics the situation where we have only one supplier with certain characteristics, and we need to determine how large an order we need to place. In the second case we try to find the optimal distribution of J. We may think of this as the scenario where we have the freedom to choose a supplier with the most "desirable" performance and place orders from it.

4.1 Fixed Distribution of $J|q$

Suppose $J|q$ follows a certain known distribution $G(q, \eta)$, where η is the vector of distribution parameters, then $\tilde{c}(J|q)$ is a univariate function of q and we can follow standard univariate optimization procedure to find q^*, i.e. the optimal q. The following two instances illustrate the point.

Example 4.1. Full delivery, i.e. $Pr(J = q|q) = 1$. We have $E(J|q) = q$ and $E(J^2|q) = q^2$. Substitute these into Eq. (3.14), we obtain:

$$\tilde{c}(J|q) = \frac{q}{2}h + \frac{K\Delta}{q}. \tag{4.1}$$

Let $\mathbb{Z}$ represent the set of all integers and $\mathbb{R}$ be the set of all real numbers. For $q \in \mathbb{R}$, denote that $\lfloor q \rfloor = \max\{m : m \leq q, m \in \mathbb{Z}\}$ and $\lceil q \rceil = \min\{m : m \geq q, m \in \mathbb{Z}\}$. Note that $\tilde{c}(J|q)$ is convex in q, so we get:

$$q^* = \arg\min_{q}\{\tilde{c}(q) : q \in \{\lfloor \tilde{q}^* \rfloor, \lceil \tilde{q}^* \rceil\}\} \quad \text{where } \tilde{q}^* = \sqrt{\tfrac{2k\Delta}{h}}. \tag{4.2}$$

Remark 4.1. Eq. (4.2) is an analog to the classical EOQ formula with the deterministic demand rate replaced by the expected net demand rate in steady state .

Example 4.2. $J|q \stackrel{d}{=} 1 + bin(q-1, \alpha)$. We have $E(J|q) = 1 + \alpha(q-1)$ and $Var(J|q) = (q-1)\alpha\beta$, where $\beta = 1 - \alpha$. Substitute these into Eq. (3.14) and simplify. We obtain:

$$\tilde{c}(J|q) = \frac{\beta h}{2} + \frac{(\alpha q + \beta)}{2} h + \frac{K\Delta - \frac{1}{2}\beta h}{\alpha q + \beta}. \tag{4.3}$$

Now consider two cases:

1. If $K \leq \frac{\beta h}{2\Delta}$, then $\tilde{c}(J|q)$ increases in q, so $q^* = 1$. Intuitively, if the fixed ordering cost is small enough, we order exactly one unit when the inventory is zero and a demand occurs.
2. If $K > \frac{\beta h}{2\Delta}$, then $\tilde{c}(J|q)$ is convex in q. Define $\hat{q}^* = \frac{1}{\alpha}\{\sqrt{\frac{2K\Delta}{h} - \beta} - \beta\}$. Then there exists a unique minimizer to $\tilde{c}(J|q)$ as follows:

$$\begin{aligned} q^* &= 1 && \text{if} \quad \hat{q}^* \leq 1, \\ q^* &= \arg\min_q \{\tilde{c}(q) : q \in \{\lfloor \hat{q}^* \rfloor, \lceil \hat{q}^* \rceil\}\} && \text{if} \quad \hat{q}^* > 1. \end{aligned} \tag{4.4}$$

Remark 4.2. In these two examples we assume η to be known. It is straightforward to extend the analysis here to the case where η is unknown and needs to be optimized. If so, $\tilde{c}(J|q)$ is a multivariate function of q and η. Instead of dealing with a univariate function we need to optimize a multivariate function.

4.2 Optimizing the Distribution of J

Let J be the delivered quantity with distribution $\alpha_i = Pr(J = i)$, $i = 1, 2, ...$ We are interested in computing the optimal probabilities $\{\alpha_1, \alpha_2, ...\}$ so as to minimize the long run average cost of Eq. (3.14) where $J|q$ is replaced by J in the function $\tilde{c}(\cdot)$. We show that the probability mass function of the optimal J concentrates on at most two points, and in many cases on a single point. To show this result, we need the following lemma which we state without proof.

Lemma 4.1. *If Y is a discrete random variable on $\mathbb{Z}$ and $E(Y) = y$, where $y \in \mathbb{R}$, then $Var(Y)$ is minimized if and only if:*

1. $Pr(Y = \lfloor y \rfloor) = \lceil y \rceil - y$,

2. $Pr(Y = \lceil y \rceil) = y - \lfloor y \rfloor$,

3. $Pr(Y = i) = 0, \forall i \in \mathbb{Z} \setminus \{\lfloor y \rfloor, \lceil y \rceil\}$.

This lemma implies a simple fact: given a fixed mean, an integer-valued discrete random variable has the minimal variance if and only if its probability mass function is only concentrated on the two integers that are closest to its mean.

Let $ber(p)$ represent a Bernoulli random variable with parameter p. Denote by J^* the optimal J and α_i^* the optimal α_i, $\forall i$. Define $\tilde{q}^* = \sqrt{\frac{2k\Delta}{h}}$ and $\Theta = K\Delta - \frac{1}{2}\lfloor \tilde{q}^* \rfloor h - \frac{1}{2}\lfloor \tilde{q}^* \rfloor^2 h$. The following theorem shows the main result of this section.

Theorem 4.1. *J^* has the following properties:*

(a) $\lfloor \tilde{q}^* \rfloor \leq E(J^*) \leq \lceil \tilde{q}^* \rceil$*;*

(b) if $\Theta < 0$, $Pr(J^* = \lfloor \tilde{q}^* \rfloor) = 1$,

if $\Theta > 0$, $Pr(J^* = \lceil \tilde{q}^* \rceil) = 1$,

if $\Theta = 0$, $J^* \stackrel{d}{=} \lfloor \tilde{q}^* \rfloor + ber(\alpha^*)$, $\forall \alpha^* \in [0, 1]$.

Proof:

Let $\hat{c}(q) = \frac{q}{2}h + \frac{K\Delta}{q}$. Suppose $E(J^*) < \lfloor \tilde{q}^* \rfloor$, then:

$$\tilde{c}(J^*) \geq \hat{c}(E(J^*)) > \hat{c}(\lfloor \tilde{q}^* \rfloor), \tag{4.5}$$

but this contradicts with the optimality of J^* since there exists J^{**} which provides a strictly smaller cost rate, where $Pr(J^{**} = \lfloor \tilde{q}^* \rfloor) = 1$. Note that in the inequality system (4.5), the first inequality stems from $Var(J^*) \geq 0$ and the second inequality is due to the convexity of $\hat{c}(q)$. Hence $E(J^*) \geq \lfloor \tilde{q}^* \rfloor$.

Similarly, suppose $E(J^*) > \lceil \tilde{q}^* \rceil$, we have:

$$\tilde{c}(J^*) \geq \hat{c}(E(J^*)) > \hat{c}(\lceil \tilde{q}^* \rceil), \tag{4.6}$$

hence $E(J^*) \leq \lceil \tilde{q}^* \rceil$. This proves property (a).

By (a) and Lemma 4.1, we know $\lfloor \tilde{q}^* \rfloor \leq E(J^*) \leq \lceil \tilde{q}^* \rceil$ and $Var(J^*)$ is minimized if and only if the probability mass are concentrated on $\lfloor \tilde{q}^* \rfloor$ and $\lceil \tilde{q}^* \rceil$. So it suffices to consider the following form for J^*:

$$J^* \stackrel{d}{=} \lfloor \tilde{q}^* \rfloor + ber(\alpha^*) \quad \text{where: } \alpha^* \in [0, 1]. \tag{4.7}$$

Now we show how to compute α^*. It is clear that $Pr(J^* = \lfloor \tilde{q}^* \rfloor) = 1 - \alpha^*$, $Pr(J^* = \lceil \tilde{q}^* \rceil) = \alpha^*$, $E(J^*) = \lfloor \tilde{q}^* \rfloor + \alpha^*$ and $Var(J^*) = \alpha^*(1 - \alpha^*)$. Substitute these into Eq. (3.14) and simplify, we obtain:

$$\tilde{c}(J^*) = \frac{\lfloor \tilde{q}^* \rfloor + \lceil \tilde{q}^* \rceil}{2} h + \frac{\Theta}{\lfloor \tilde{q}^* \rfloor + \alpha^*}. \tag{4.8}$$

Notice that $\tilde{c}(J^*)$ is a monotone function on α^* and hence property (b) follows. ■.

Remark 4.3. It is interesting to see that in most cases J^* is deterministic rather than stochastic and it is optimal for us to place any size of an order which guarantees that the property (b) in theorem 4.1 holds. Observe the connection between theorem 4.1 and example 4.1: recall Eq. (4.2) in example 4.1, one can show that $q^* = \lfloor \tilde{q}^* \rfloor$ if $\Theta < 0$, $q^* = \lceil \tilde{q}^* \rceil$ if $\Theta > 0$ and $q^* \in \{\lfloor \tilde{q}^* \rfloor, \lceil \tilde{q}^* \rceil\}$ if $\Theta = 0$.

5 Applications to Warranty Inventory Management

In this section we apply the results obtained above to an integrated warranty inventory management model under continuous review. We introduce the model setup first and then discuss the main results.

5.1 Warranty Inventory Management Model

In this section we consider an inventory model where demands may come from two sources: (1) sales of products and (2) replacement request from a failed product under warranty. We assume a Free-Replacement warranty. This type of inventory model is of great significance in industries, especially in those that produce durable products sold with warranties on them, e.g. vehicle batteries, electric appliance, computer components etc.

We concentrate first on the case when the demand only comes from failure items under warranty. The model is specified as follows. We assume that the sales process follow a $PP(\lambda)$. Product lifetimes are i.i.d $PH(\alpha, T)$, a phase-type random variable with parameters (α, T), with m phases. Warranty periods of each item are i.i.d. $PH(\beta, W)$ with n phases. The notation of a phase-type random variable is as in Kulkarni [4]. The sales process, product lifetimes and warranty periods are mutually independent. Each item that fails while under warranty is replaced with a new one from the inventory. Note that the warranty may be renewing or non-renewing (see Blischke and Murthy [1]), which results in different warranty period distribution of the replaced product. We consider these two cases separately. We assume there is *no* leadtime in replenishing inventory *nor* backlogging in the inventory system. All demands are satisfied.

The cost structure is as follows. Procurement cost is c per item and fixed ordering cost is K. Holding cost rate is h. The objective is to minimize the long run average cost. Since there is *no* leadtime, events are specified to happen in the following sequence: demand realized, inventory reviewed, order placed, inventory replenished, demand satisfied, and cost assessed. Note that the order placed is received instantaneously and can be used to satisfy the demand immediately.

Inventory control under fluctuating demand environment has been studied in literatures before. For instance, Song and Zipkin [11] consider a general setup of this problem where backlogging and random lead times are allowed. They model the problem as a dynamic programming one and show that the optimal policy for a problem with fixed ordering cost is an environment-dependent (r, q) policy, i.e. the reorder point r and the size of order q dependent on the state of the external environment at the time of ordering. However, in reality it is usually difficult to tell what the current state of the external environment is. To compute the optimal policy is also quite tedious as suggested in [11]. Hence it appears that the optimal policy is not easy to compute or to implement. Instead, we can try to find the optimal policy in the (environment-independent) (r, q) policy class, which works as follows, as soon as the inventory level hits r we immediately order q unit of products. Here we assume all q units are effective. Since the leadtimes are zero in our model, it is easy to see that the optimal (r, q) policy here has $r = -1$.

5.2 Main Results

Let $Y_{ij}(t)$ be the number of products with lifetime in phase i and warranty period in phase j at time t and $Y(t) = [Y_{ij}(t)]$ be a $m \times n$ matrix. One can easily see that $\{Y(t), t \geq 0\}$ is a CTMC and this inventory model is a special case of the model with jumps as discussed in section 3 with $\{Y(t), t \geq 0\}$ serving the role of the external environment. In particular, the production rate here is zero at all time $t \geq 0$ and for all state $Y(t)$. Let $X(t)$ represent the inventory level at time t and $(X, Y) \stackrel{d}{=} \lim_{t\to\infty}(X(t), Y(t))$. From theorem 3.2 and remarks 4.2 and 4.3, we have the following results directly:

Theorem 5.1. (X, Y) *has the following properties:*

1. X *and* Y *are independent,*

2. X *is a discrete uniform random variable on* $\{0, 1, ..., q - 1\}$.

The next lemma gives a closed form solution for the limiting distribution of $Y(t)$. We study two cases: renewing and non-renewing warranty. Let $P(\gamma)$ represent a Poisson random variable with mean γ. Define $\mu_{ij} = -T_{ii} - W_{jj}$, $\forall i, j$ and $T^0 = -Te$ to be a $1 \times m$ row vector whose ith entry is T_i^0.

Lemma 5.1. *For renewing warranty,* Y *has the following properties:*

1. Y_{ij} *is independent with* Y_{st} *if* $(i, j) \neq (s, t)$*,* *2.* $Y_{ij} \stackrel{d}{=} P(\frac{a_{ij}}{\mu_{ij}})$*, where* a_{ij} *solves the following linear systems:*

$$a_{ij} = \alpha_i \beta_j \lambda + \sum_{k \neq j} \frac{w_{kj}}{\mu_{ik}} a_{ik} + \sum_{k \neq i} \frac{t_{ki}}{\mu_{kj}} a_{kj} + \sum_{x,y} \frac{T_x^0}{\mu_{xy}} \alpha_x \beta_y a_{xy} \quad \forall i, j. \tag{5.1}$$

For non-renewing warranty, Y *has the following properties:*

1. Y_{ij} *is independent with* Y_{st} *if* $(i,j) \neq (s,t)$*,*

2. $Y_{ij} \stackrel{d}{=} P(\frac{a_{ij}}{\mu_{ij}})$*, where* a_{ij} *solves the following linear systems:*

$$a_{ij} = \alpha_i \beta_j \lambda + \sum_{k \neq j} \frac{w_{kj}}{\mu_{ik}} a_{ik} + \sum_{k \neq i} \frac{t_{ki}}{\mu_{kj}} a_{kj} + \sum_{k} \frac{T_k^0}{\mu_{kj}} \alpha_i a_{kj} \quad \forall i, j. \tag{5.2}$$

Proof:

We prove the results for the renewing warranty case and the non-renewing warranty case follows a similar proof.

For the renewing case, it is not difficult to see that the process $\{Y(t) = [Y_{ij}(t)], t \geq 0\}$ behaves as a Jackson Network with $m \times n$ stations and infinite servers at each station. In terms of Jackson Network settings, $Y_{ij}(t)$ represents the queue length of station (i,j) at time t. The service rate of a server at station (i,j) is μ_{ij}. The external arrival rate to station (i,j) is $\alpha_i \beta_j \lambda$. When a customer completes service at station (i,j), he joins the queue at station (i,k) with probability $\frac{w_{jk}}{\mu_{ij}} + \frac{T_i^0}{\mu_{ij}} \alpha_i \beta_k$, joins the queue at station (k,j) with probability $\frac{t_{ik}}{\mu_{ij}} + \frac{T_i^0}{\mu_{ij}} \alpha_k \beta_j$, joins the queue at station (x,y) where $x \neq i$ and $y \neq j$ with probability $\frac{T_i^0}{\mu_{ij}} \alpha_x \beta_y$ and departs the system with probability $\frac{W_j^0}{\mu_{ij}}$ where W_j^0 is the jth entry of the $1 \times n$ row vector $W^0 = -We$. So a_{ij} represents the total arrival rate to station (i,j). Then the results follow directly. See theorem 7.5 in Kulkarni [4] for more details. ■.

Notice that T_i^0 is the failure rate of a product with lifetime in phase i. Then at time t the demand rate in state $Y(t)$ can be computed as $\mu_{Y(t)} = \sum_{i,j} Y_{ij}(t) T_i^0$, $\forall t$. Let $\Delta = \sum_{i,j} \frac{a_{ij}}{\mu_{ij}} T_i^0$, $\hat{c}(q) = \frac{q}{2}h + \frac{K\Delta}{q}$ and $\tilde{q}^* = \sqrt{\frac{2K\Delta}{h}}$. By example 4.1, theorem 5.1 and lemma 5.1, we have the following result:

Theorem 5.2. *The optimal order quantity q^* can be computed as follows:*

$$q^* = \arg\min_q \{\hat{c}(q) : q \in \{\lfloor \tilde{q}^* \rfloor, \lceil \tilde{q}^* \rceil\}\}. \tag{5.3}$$

Proof:

It remains to verify that the mean demand rate $\Delta = \sum_{i,j} \frac{a_{ij}}{\mu_{ij}} T_i^0$. This follows as a special case of Eq. (3.9), from which and lemma 5.1 we have:

$$\begin{aligned} \Delta &= \sum_{i,j,k} k Pr[Y_{ij} = k] T_i^0 \\ &= \sum_{i,j} \frac{a_{ij}}{\mu_{ij}} T_i^0. \end{aligned} \tag{5.4}$$

■.

Notice that T_i^0 is the demand rate for a replacement if a product's lifetime is in phase i and $\frac{a_{ij}}{\mu_{ij}}$ is the long run average number of products with lifetime in phase i and warranty in phase j. So Δ represents the total demand rate for replacements at steady state. One may observe that Δ is directly proportional to the arrival rate λ. The next theorem clarifies this point.

Theorem 5.3. *For renewing warranty, $\Delta = \lambda \sum_{i,j} \alpha_i \beta_j f_{ij}$, where f_{ij} solves the following linear systems:*

$$f_{ij} = \frac{T_i^0}{\mu_{ij}} + \sum_{j \neq k} \frac{w_{jk}}{\mu_{ij}} f_{ik} + \sum_{i \neq k} \frac{t_{ik}}{\mu_{ij}} f_{kj} + \frac{T_i^0}{\mu_{ij}} \sum_{x,y} \alpha_x \beta_y f_{xy} \quad \forall i, j. \tag{5.5}$$

For non-renewing warranty, $\Delta = \lambda \sum_{i,j} \alpha_i \beta_j f_{ij}$, where f_{ij} solves the following linear systems:

$$f_{ij} = \frac{T_i^0}{\mu_{ij}} + \sum_{j \neq k} \frac{w_{jk}}{\mu_{ij}} f_{ik} + \sum_{i \neq k} \frac{t_{ik}}{\mu_{ij}} f_{kj} + \frac{T_i^0}{\mu_{ij}} \sum_{k} \alpha_k f_{kj} \quad \forall i, j. \tag{5.6}$$

Proof:

Use $\otimes$ to represent the Kronecker product operator. Let $a = [a_{1,\cdot}, a_{2,\cdot}, ..., a_{m,\cdot}]^T$, $f = [f_{1,\cdot}, f_{2,\cdot}, ..., f_{m,\cdot}]^T$, $u = \lambda \alpha^T \otimes \beta^T$ and $v = [v_1, v_2, ..., v_{m \times n}]^T$ where $v_{(i-1)n+j} = \frac{T_i^0}{\mu_{ij}}$. Note that a, f, u and v are all collum vectors with dimension $m \times n$. For renewing warranty case, we can write systems (5.1) and (5.5) into matrix form as follow:

$$\begin{aligned} Ga &= u, \\ Hf &= v, \end{aligned} \tag{5.7}$$

where G is the coefficient matrix of a in system (5.1) and H is the coefficient matrix of f in system (5.5). It is tedious but straightforward to verify that $G^T = H$, hence $a^T v = f^T u$. This proves the results for the renewing warranty case. Similarly, we can prove the non-renewing warranty case. ■.

Remark 5.1. f_{ij} represents the expected replacements for a new product starting with lifetime in phase i and warranty in phase j before its warranty expires, so $\sum_{i,j} \alpha_i \beta_j f_{ij}$ is the expected replacements for a new product during its warranty period.

Remark 5.2. In theorem 5.3, Δ is the total expected failure rate computed from the perspective of sales process. In the long run, sales per unit of time is λ and per product requires $\sum_{i,j} \alpha_i \beta_j f_{ij}$ replacements on average. So in the long run we require $\Delta = \lambda \sum_{i,j} \alpha_i \beta_j f_{ij}$ replacements per unit of time. However, in theorem 5.2, we look at all products under warranty in steady state, examine their failure rates accordingly and compute Δ directly. So intuitively these two methods should give the same results for Δ and theorem 5.3 verifies this point.

Remark 5.3. Since the set of phase-type distributions is dense in the set of all distributions over $[0, +\infty)$, theorem 5.3 implies that if the product life time and warranty period follow two independent general distributions over $[0, +\infty)$, then $\Delta = \lambda f$ where f is the expected replacements for a sale of new product. Ja [3] provides the method to compute f for both renewing and non-renewing warranty cases. So we can compute Δ in this way rather than going through a full analysis of the whole system as done in theorem 5.1 and lemma 5.1. With Δ, we can easily calculate q^* according to theorem 5.2.

5.3 Extensions

In this section we discuss two extensions to the warranty inventory model discussed in section 5.1. One extension is to relax the sales process to a renewal process. The other extension is to consider an inventory model facing a combined demand which comes from both sales and replacement requests of failed products under warranty.

5.3.1 General Sales Process

Instead of assuming that the sales process follows a $PP(\lambda)$, we generalize it to be a renewal process with inter-arrival time distributed as a phase-type random variable, denoted as $PH(\gamma, D)$ with l phases. Let $S(t)$ denote the current phase of an inter-arrival time during the sales process, $Y(t)$ be defined in the same way as in section 5.1 and $Z(t) = (S(t), Y(t))$. One can easily see that this inventory model is also a special case of the model with jumps as discussed in section 3 where $\{Z(t), t \geq 0\}$ is a CTMC and serves the role of the external environment. Let $X(t)$ represent the inventory level at time t and $(X, Z) \stackrel{d}{=} \lim_{t\to\infty}(X(t), Z(t))$, then we immediately have an analog theorem to theorem 5.1 as follows:

Theorem 5.4. (X, Z) *has the following properties:*

1. X *and* Z *are independent,*

2. X *is a discrete uniform random variable on* $\{0, 1, ..., q-1\}$.

However, it does not seem to be an easy task to analyze $\{Z(t), t \geq 0\}$ process as we do in lemma 5.1 to obtain the closed form solution for the steady state distribution. Instead, similar to theorem 5.3, the next one gives a method to compute q^* without analyzing

$\{Z(t), t \geq 0\}$ process explicitly. Let $\lambda = -\frac{1}{\gamma D^{-1} e}$ and $\Delta = \lambda \sum_{i,j} \alpha_i \beta_j f_{ij}$ and recall that $\hat{c}(q) = \frac{q}{2}h + \frac{K\Delta}{q}$ and $\tilde{q}^* = \sqrt{\frac{2K\Delta}{h}}$.

Theorem 5.5. *The optimal order quantity q^* can be computed as follows:*

$$q^* = \arg\min_q \{\hat{c}(q) : q \in \{\lfloor \tilde{q}^* \rfloor, \lceil \tilde{q}^* \rceil\}\}. \tag{5.8}$$

Proof:

Let $R(t)$ represent the total number of replacements up to time t. By Eq. (3.9), we have:

$$\lim_{t \to \infty} \frac{R(t)}{t} = \Delta = \sum_{i,j,k,w} k Pr[(S, Y_{ij}) = (w, k)] T_i^0, \tag{5.9}$$

where $(S, Y) \stackrel{d}{=} \lim_{t \to \infty}(S(t), Y(t)) = \lim_{t \to \infty} Z(t)$.

It remains to show that $\lim_{t \to \infty} \frac{R(t)}{t} = \lambda \sum_{i,j} \alpha_i \beta_j f_{ij}$. Let $N'(t)$ represent the total sales up to time t. Denote R'_i as the number of replacements for the ith sale and T'_i as the ith inter-sale time. Let $R'(t) = \sum_{i=1}^{N'(t)} R'_i$. By definition of $R(t)$ and $R'(t)$ we know:

$$\lim_{t \to \infty} \frac{R(t)}{t} = \lim_{t \to \infty} \frac{R'(t)}{t}. \tag{5.10}$$

It is not difficult to see that $\{R'(t), t \geq 0\}$ is a renewal reward process, so:

$$\lim_{t \to \infty} \frac{R'(t)}{t} = \frac{E(R'_1)}{E(T'_1)} = \frac{\sum_{i,j} \alpha_i \beta_j f_{ij}}{-\gamma D^{-1} e}. \tag{5.11}$$

This proves our results. ■.

Remark 5.4. Theorem 5.5 formalizes the intuition given in remark 5.2. It provides an alternative and more general way to prove theorem 5.3.

Remark 5.5. Similar to remark 5.3, we see that theorem 5.5 can be extended to a general warranty inventory system where the sales process is a renewal process and the product life time and warranty period follow two independent general distributions over $[0, +\infty)$.

5.3.2 An Inventory Model Facing a Combined Demand

It is straightforward to accommodate a warranty inventory model facing a combined demand. Theorem 5.4 still applies since the external environment process is still a CTMC and no production occurs at any state at any time. In light of theorem 5.5, instead of R_i' units of demands, the ith sale incurs $R_i' + 1$ units of demands to the inventory. So one just need to replace f_{ij} by $f_{ij} + 1$ in theorem 5.5 for the combined demand model.

6 Conclusion

In this paper we study a production-inventory system where the production and demand are Poisson processes and their rates are modulated by an external environment. Whenever a production or demand occurs, the state of the environment may switch to another state instantaneously (or remain unchanged) with a given probability. Between two consecutive switches the environment process behaves as a CTMC. We consider models without jumps and with jumps, and show an interesting decomposition property between the limiting distributions of the inventory level in these two models. We derive the optimal order quantity which follows the classical EOQ formula with the deterministic demand rate replaced by the expected demand rate in steady state. Then we present an application of these results to a warranty inventory control problem.

Although the jump size in our model is random, it is independent with the state of the environment process prior to the jump. It will be interesting to study the problem where the jump size depends on the environment process. Another possible extension to the warranty inventory model will be to consider a positive leadtime between the placement and arrival of an order and/or allow backlogging in the inventory management. But these two extensions break down the decomposition property and hence make the problem substantially more difficult. We may also consider the problem where the production is under our control. However, a proper cost measure is necessary before one can attempt this extension.

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