ENUMERATION IN CONVEX GEOMETRIES AND ASSOCIATED POLYTOPAL SUBDIVISIONS OF SPHERES

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Abstract. We construct CW spheres from the lattices that arise as the closed sets of a convex closure, the meet-distributive lattices. These spheres are nearly polytopal, in the sense that their barycentric subdivisions are simplicial polytopes. The complete information on the numbers of faces and chains of faces in these spheres can be obtained from the defining lattices in a manner analogous to the relation between arrangements of hyperplanes and their underlying geometric intersection lattices.

1. Introduction

A well known result due to Zaslavsky [28] shows that the numbers of faces in an arrangement of hyperplanes in a real Euclidean space can be read from the underlying geometric lattice of all intersections of these hyperplanes. This result was extended to the determination of the numbers of chains of faces in arrangements in [2, 9]. In [5], the numbers of chains in an arrangement were shown to depend only on the numbers of chains in the associated geometric lattice. A particularly simple form of this relationship, in terms of quasisymmetric functions, was given in [3].

Geometric lattices (matroids) are combinatorial abstractions of linear span in vector spaces. There is a different combinatorial model for convex span, known as convex geometries (or anti-matroids) [12, 13, 14, 15], for which the corresponding lattices are the meet-distributive lattices. We show here that a similar situation exists for these; that is, for

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each convex geometry, we construct a regular CW sphere, whose enumerative properties are related to those of the underlying geometry in essentially the same way. Moreover, these spheres are nearly polytopes, in the sense that their first barycentric subdivisions are combinatorially simplicial convex polytopes.

We begin by establishing some notation. Our basic object of study is a combinatorial closure operation called a *convex* or *anti-exchange* closure. This is defined on a finite set, which we will take, without loss of generality, to be the set $[n] := \{1, 2, \ldots, n\}$.

Definition 1.1. A *convex closure* is a function $\langle\cdot\rangle : 2^{[n]} \to 2^{[n]}$, $A \mapsto \langle A\rangle$, such that, for $A, B \subset [n]$,

(1) $A \subset \langle A\rangle$
(2) if $A \subset B$ then $\langle A\rangle \subset \langle B\rangle$
(3) $\langle A\rangle = \langle\langle A\rangle\rangle$
(4) if $x, y \notin \langle A\rangle$ and $x \in \langle A \cup y\rangle$ then $y \notin \langle A \cup x\rangle$.

The last condition is often called the *anti-exchange* axiom, and the complements of the closed sets of such a closure system has been called an *anti-matroid.* We will call a set together with a convex closure operator on it a *convex geometry.* The set of *closed sets* of a convex geometry, that is, those sets A satisfying $A = \langle A\rangle$, form a lattice when ordered by set inclusion. Such lattices are precisely the meet-distributive lattices. A lattice L is *meet-distributive* if for each $y \in L$, if $x \in L$ is the meet of (all the) elements covered by y, then the interval $[x, y]$ is a boolean algebra.

One example of an anti-exchange closure operator is ideal closure on a partially ordered set P; here, for $A \subset P$, $\langle A\rangle$ denotes the (lower) order ideal generated by A. The lattices of closed sets of these are precisely the distributive lattices. Another class of examples comes from considering convex closure on a finite point set in Euclidean space. Figure 1 illustrates the convex geometry formed by three collinear points a, b, c. Note that the set $\{a, c\}$ is not closed since its closure is $\{a, b, c\}$.

The theory of convex geometries and meet-distributive lattices was extensively developed about 20 years ago in a series of papers by Edelman and coauthors [12, 13, 14, 15]. See also [20] and [11] for general discussions. An important (and characterizing) property of convex geometries is that every set has a unique minimal generating set, that is, for each $A \subset [n]$, there is a unique minimal subset $ext(A) \subset A$ so that $\langle A\rangle = \langle ext(A)\rangle$ [13, Theorem 2.1]. The elements of $ext(A)$ are called the *extreme points* of A.

A *simplicial complex* on a finite set V is a family of subsets $\Delta \subset 2^V$ such that if $\tau \subset \sigma \in \Delta$ then $\tau \in \Delta$ and $\{v\} \in \Delta$ for all $v \in V$. The

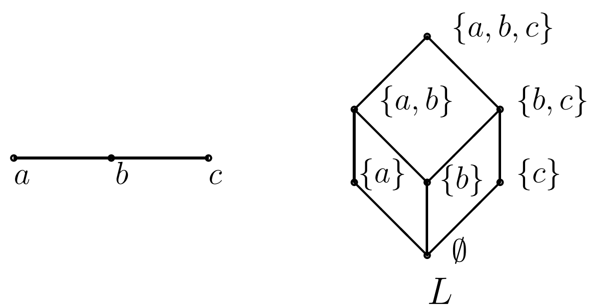

FIGURE 1. The convex geometry of three collinear points and its associated meet-distributive lattice L.

elements of Δ are called the *faces* of the complex and the elements of V are its *vertices*. We will need an operation on simplicial complexes known as *stellar subdivision*.

Definition 1.2. The *stellar subdivision* of a simplicial complex Δ over a nonempty face $\sigma \in \Delta$ is the simplicial complex $sd_\sigma(\Delta)$ on the set $V \cup \{v_\sigma\}$, where v_σ is a new vertex, consisting of

(1) all $\tau \in \Delta$ such that $\tau \not\supset \sigma$, and
(2) all $\tau \cup \{v_\sigma\}$ where $\tau \in \Delta$, $\tau \not\supset \sigma$ and $\tau \cup \sigma \in \Delta$.

The use of stellar subdivision to describe order complexes of posets was begun in [21], where it was shown that the order complex of any distributive lattice can be obtained from a simplex by a sequence of stellar subdivisions. Although this result and some of its implications were discussed in [22], its proof was never published. We will give a generalization of this result to meet-distributive lattices in the next section. The proof is an adaptation of that in [21].

More recently, stellar subdivision was used in [10] to produce the order complex of a so-called Bier poset of a poset P from the order complex of P.

In §2, we describe the order complex of a meet-distributive lattice as a stellar subdivision of a simplex. We use this in §3 to construct the sphere associated with the lattice. Finally, in §4 we relate enumeration in this sphere to that of the lattice.

2. Order complexes of meet-distributive lattices

Let L be an arbitrary meet-distributive lattice. We can assume L is the lattice of closed sets of a convex closure $\langle\cdot\rangle$ on the set $[n]$, for some $n > 0$. L has unique maximal element $\hat{1} = \langle [n] \rangle$ and minimal element $\hat{0} = \langle \emptyset \rangle$ (we may assume $\langle \emptyset \rangle = \emptyset$, although this will not be important here). For simplicity of notation, we will write $\langle i\rangle$ for the *principal* closed set $\langle\{i\}\rangle$ whenever $i \in [n]$. These are precisely the *join-irreducible* elements of L, that is, those $x \in L \setminus \hat{0}$ that cannot be written as $y \vee z$, with $y, z < x$. (This follows, for example, from [13, Theorem 2.1(f)].)

In fact, the convex closure $\langle\cdot\rangle$ is uniquely defined from the lattice L: we take $[n]$ to be an enumeration of the join-irreducible elements of L and define, for $A \subset [n]$,

$$\langle A\rangle = \left\{ j \in [n] \;\middle|\; j \leq \bigvee_{i \in A} i \right\}.$$

Thus we are free, without loss of generality, to use the closure relation when making constructions concerning the lattice L.

Note that $\Delta(L\setminus\hat{0})$, the order complex of $L\setminus\hat{0}$, is a simplicial complex on the vertex set $V = \{\langle A\rangle \,|A \subset [n], A \neq \emptyset\}$. Consider the simplex of all principal closed sets (join-irreducibles) $\{\langle i\rangle \;\mid i \in [n]\}$, and let Δ_0 be the complex consisting of this simplex and all its faces (subsets).

Theorem 2.1. *For any meet-distributive lattice L, $\Delta(L \setminus \hat{0})$ can be obtained from the simplex of join-irreducible elements by a sequence of stellar subdivisions.*

Proof. Suppose L is the lattice of closed sets of a convex closure $\langle\cdot\rangle$ on $[n]$. Let $X_1, X_2, \ldots, X_k$ be a reverse linear extension of $L \setminus \hat{0}$, that is, the X_i are all the nonempty closed subsets in $[n]$, ordered so that we never have $X_i \subset X_j$ if $i < j$. In particular, $X_1 = [n]$.

The order complex $\Delta(L \setminus \hat{0})$ can be obtained from Δ_0 by a sequence of stellar subdivisions as follows. For $i = 1, \ldots, k$, let

$$\Delta_i = sd_{ext(X_i)}(\Delta_{i-1}),$$

where, by a slight abuse of notation, $ext(X_i)$ will denote the face of Δ_{i-1} having vertices $\langle i\rangle$, $i \in ext(X_i)$. The new vertex added at the i^{th} step will be denoted simply by X_i. Note that because of the ordering of the X_i, the face $ext(X_i)$ is in the complex Δ_{i-1}, so each of these subdivisions is defined.

We claim that $\Delta_k = \Delta(L \setminus \hat{0})$. The proof proceeds by induction on n. The case $n = 1$ is clear.

When $n > 1$, consider the complex $\Delta_1 = sd_{ext(X_1)}(\Delta_0)$. Since Δ_0 is a simplex, the new vertex X_1 is a cone point, that is, it is in every maximal simplex. The base of this cone (the *link* of X_1) consists of all the facets $F_1, \ldots, F_m$ of Δ_0 that are opposite to vertices in $ext(X_1)$. By relabeling if necessary, we can assume that $ext(X_1) = \{1, 2, \ldots, m\}$, and $F_i = \{\langle j \rangle \mid j \neq i\}$. Since all further subdivisions are made on faces not containing X_1, the vertex X_1 remains a cone point in all Δ_i. So it is enough to consider the effect of further subdivisions on each of the facets F_i.

Now, by induction, the face F_i is subdivided so that it becomes the order complex of $L_i \setminus \hat{0}$, where L_i is the lattice of closed subsets of $[n] \setminus \{i\}$. Since X_1 is a cone point in Δ_k, and X_1 is in every maximal chain in L, it follows that Δ_k is the order complex of $L \setminus \hat{0}$. □

Notice that the stellar subdivisions over the join-irreducibles are redundant and can be omitted without loss. Figure 2 give the sequence of subdivisions leading to the order complex of the meet-distributive lattice generated by the example of three collinear points.

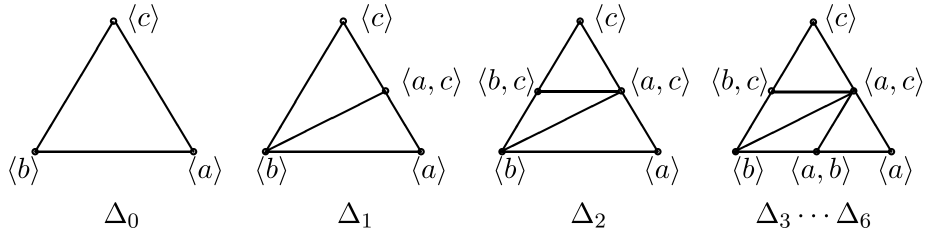

FIGURE 2. The sequence of Δ_i for the example in Figure 1.

By a *polyhedral ball* we will mean a simplicial complex that is topologically a d-dimensional ball and can be embedded to give a regular triangulation, that is, one that admits a strictly convex piecewise-linear function (see, for example, [6] for the definitions). Polyhedral balls are known to satisfy strong enumerative conditions [7].

Corollary 2.2. *For any meet-distributive lattice $L \neq B_n$, the order complex $\Delta(L \setminus \hat{0}, \hat{1})$ is a polyhedral ball.*

Proof. Let L be the lattice of closed sets of the convex closure $\langle \cdot \rangle$ on $[n]$. Since $L \neq B_n$, we have that $ext([n]) \neq [n]$, and so every stellar subdivision that is involved in producing $\Delta(L \setminus \hat{0})$ takes place on the boundary of the simplex Δ_0.

Since being the boundary complex of a simplicial convex polytope is preserved under taking stellar subdivisions [22], we conclude that the boundary of $\Delta(L \setminus \hat{0})$ is the boundary of a simplicial convex polytope Q. By means of a projective transformation that sends the vertex $X_1 = [n] = \hat{1}$ to the point at infinity, we see that the image of Q under such a map is the graph of a strictly convex function over $\Delta(L \setminus \hat{0}, \hat{1})$. □

It was shown in [22] that stellar subdivision preserves the property of being *vertex decomposable*, which in turn implies shellability. As a consequence we get that both $\Delta(L \setminus \hat{0})$ and $\Delta(L \setminus \hat{0}, \hat{1})$ are vertex decomposable and hence shellable, as stated in [8, Theorem 8.1] (and its proof) in the language of greedoids. Theorem 2.1 was first proved for distributive lattices in [21] precisely to show that order complexes of distributive lattices were shellable. The result in [21] was stated for $\Delta(L)$, which is a cone over the complex we consider.

3. The associated CW spheres

We define now a triangulated sphere derived from the order complex of a meet-distributive lattice L. It will turn out that this triangulated sphere is the barycentric subdivision of a regular CW sphere that has the same enumerative relationship to L^* (the dual to L) as an arrangement of hyperplanes (oriented matroid) has to the underlying geometric lattice.

3.1. **The complex $\pm\Delta$.** For a meet-distributive lattice L, let $\Delta = \Delta(L \setminus \hat{0})$, a triangulation of the $(n-1)$-simplex Δ_0. We will define a triangulation $\pm\Delta$ of the n-dimensional crosspolytope O_n as follows. If the vertex set of the simplex is $[n] = \{1, 2, \ldots, n\}$, then that of the crosspolytope is $\pm[n] = \{\pm 1, \pm 2, \ldots, \pm n\}$. Faces of the crosspolytope are all $\sigma \subset \pm[n]$ such that not both i and $-i$ are in σ.

We reflect the triangulation Δ to obtain a triangulation $\pm\Delta$ of the crosspolytope, much as the crosspolytope can be built by reflecting the simplex generated by the unit vectors. We consider the simplex Δ_0 to be embedded as the convex hull of the unit vectors and define the triangulation $\pm\Delta$ by reflecting the triangulation Δ. Formally, $\pm\Delta$ is the simplicial complex whose vertices are all pairs (A, ε), where $A \in L \setminus \hat{0}$ and ε is a map from $ext(A)$ to $\{\pm 1\}$. For arbitrary $\varepsilon : [n] \to \{\pm 1\}$ and $\sigma = \{A_1, A_2, \ldots, A_k\} \in \Delta$, let

$$\sigma^\varepsilon := \big\{(A_1, \varepsilon|_{ext(A_1)}), (A_2, \varepsilon|_{ext(A_2)}), \ldots, (A_k, \varepsilon|_{ext(A_k)})\big\}$$

and

$$\Delta^\varepsilon = \big\{\sigma^\varepsilon | \sigma \in \Delta\big\}.$$

Δ^ε is essentially the triangulation Δ transferred to the face of the crosspolytope given by the sign pattern ε. Finally, define

$$\pm\Delta = \bigcup_\varepsilon \Delta^\varepsilon,$$

the union being taken over all $\varepsilon : [n] \to \{\pm 1\}$.

Remark 3.1. Note that boundary faces of Δ can result in faces of $\pm\Delta$ having more than one name; in fact, $\sigma^\varepsilon = \sigma^\rho$ if and only if ε and ρ agree on the set

$$ext(\sigma) := \bigcup_{i=1}^{k} ext(A_i).$$

Figure 3 shows the complex $\pm\Delta$ for the example of three collinear points.

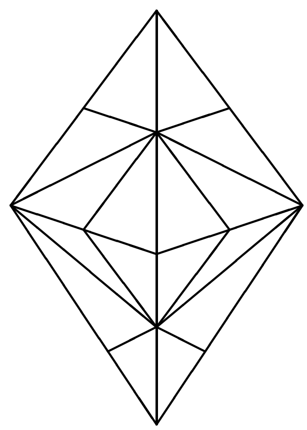

FIGURE 3. Triangulation of the boundary of the octahedron induced by reflecting Δ_6.

Theorem 3.2. *For any meet-distributive lattice L, $\pm\Delta(L \setminus \hat{0})$ can be obtained from the n-dimensional crosspolytope by a sequence of stellar subdivisions, and so it is combinatorially the boundary complex of an n-dimensional simplicial polytope, where n is the number of join-irreducibles of L.*

Proof. As before, suppose L is the lattice of closed sets of a convex closure $\langle\cdot\rangle$ on $[n]$, and let $X_1, X_2, \ldots, X_k$ be a reverse linear extension of $L \setminus \hat{0}$. We can extend this order to an order on all pairs (X_i, ε), $\varepsilon : ext(X_i) \to \{\pm 1\}$, by ordering lexicographically, with the given order on the first coordinate and any order on the second.

It is now relatively straightforward to adapt the proof of Theorem 2.1 to show that the complex $\pm\Delta$ is obtained by carrying out stellar subdivisions over faces of O_n in the order given by the order of the (X_i, ε). The subdivision corresponding to (X_i, ε) is done over the face $\{\varepsilon(j) \cdot j \mid j \in ext(X_i)\}$ of O_n; in $\pm\Delta$, every face containing $\{\varepsilon(j) \cdot j \mid j \in ext(X_i)\}$ is subdivided as it would be by doing the subdivision in the boundary of O_n. Again, since stellar subdivision preserves the property of being the boundary complex of a polytope, the result follows. □

3.2. **The poset Q_L.** We construct a regular CW complex Σ having $\pm\Delta$ as its barycentric subdivision. Equivalently, if $\mathcal{F}(\Sigma)$ is the face poset of Σ, then $\Delta(\mathcal{F}(\Sigma)) = \pm\Delta$.

We begin by defining a poset Q_L associated to any meet-distributive lattice L. The elements of Q_L are all pairs (A, ε), where $A \in L$ and ε is a map from $ext(A)$ to $\{\pm 1\}$. We define the order relation on Q_L by $(A, \varepsilon) \leq (B, \delta)$ if and only if $A \subseteq B$ and the maps ε, δ agree on the set $ext(A) \cap ext(B)$. We include an element $\hat{1} \in Q_L$ for convenience; the element $(\hat{0}, \emptyset)$ corresponding to $\hat{0} \in L$ serves as $\hat{0}$ in Q_L. Note that when L is distributive, Q_L is the signed Birkhoff poset of [19].

Proposition 3.3. $\Delta(Q_L \setminus \hat{0}, \hat{1}) = \pm\Delta$

Proof. The maximal simplices in $\pm\Delta$ are the simplices

$$\sigma^\varepsilon := \{(A_1, \varepsilon|_{ext(A_1)}), (A_2, \varepsilon|_{ext(A_2)}), \ldots, (A_n, \varepsilon|_{ext(A_n)})\},$$

where $A_1 \subset A_2 \subset \cdots \subset A_n$ is a maximal chain in $L \setminus \hat{0}$. Then clearly,

$$(A_1, \varepsilon|_{ext(A_1)}) < (A_2, \varepsilon|_{ext(A_2)}) < \cdots < (A_k, \varepsilon|_{ext(A_n)})$$

is a maximal chain in $Q_L \setminus \hat{0}, \hat{1}$.

Conversely, if

$$(A_1, \varepsilon_1) < (A_2, \varepsilon_2) < \cdots < (A_n, \varepsilon_n)$$

is a maximal chain in $Q_L \setminus \hat{0}, \hat{1}$, then, if we let $\sigma = \{A_1, A_2, \ldots, A_n\} \in \Delta$, we have $ext(\sigma) = [n]$ and so there is an $\varepsilon : [n] \to \{\pm 1\}$ such that $\varepsilon_i = \varepsilon|_{ext(A_i)}$ for each i. Thus

$$\{(A_1, \varepsilon_1), (A_2, \varepsilon_2), \ldots, (A_n, \varepsilon_n)\} = \sigma^\varepsilon$$

is a maximal simplex in $\pm\Delta$. □

Next, we define a cell complex Σ_L from the lattice L (the underlying convex closure $\langle\cdot\rangle$ on $[n]$ and the simplicial complex $\pm\Delta$) as follows. For each $A \in L \setminus \hat{0}$ and $\varepsilon : ext(A) \to \{\pm 1\}$, we define a cell $C_{(A,\varepsilon)}$ that is a union of simplices in $\pm\Delta$. For $A = [n]$, we take $C_{(A,\varepsilon)}$ to be the star of (A, ε) in the complex $\pm\Delta$, that is, the union of all maximal simplices containing the vertex (A, ε).

For proper closed sets $A \in L$, we consider the subgeometry $\langle\cdot\rangle$ restricted to subsets of A, with lattice $L_A = [\hat{0}, A]$ and order complex $\Delta_A = \Delta(L_A \setminus \hat{0})$. The complex Δ_A is the subcomplex of Δ restricted to the face defined by $x_i = 0$ for $i \notin A$, and the corresponding complex $\pm\Delta_A$ is the subcomplex of $\pm\Delta$ obtained by intersecting with the subspace defined by $x_i = 0$ for $i \notin A$. For any $A \in L$ and any $\varepsilon : ext(A) \to \{\pm 1\}$, we define the cell $C_{(A,\varepsilon)}$ to be the star of (A, ε) in the complex $\pm\Delta_A$.

Since $\pm\Delta_A$ is the boundary of a simplicial polytope by Theorem 3.2, each cell $C_{(A,\varepsilon)}$ is topologically a disk of dimension $|A| - 1$, and its boundary is the link of the vertex (A, ε) in the complex $\pm\Delta_A$ and so is a sphere. We define Σ_L to be the collection of all the cells $C_{(A,\varepsilon)}$, $A \in L \setminus \hat{0}$.

Lemma 3.4. *The boundary of $C_{(A,\varepsilon)}$ is the union of all cells $C_{(B,\delta)}$, where $B \subset A, B \neq A$ and the maps ε, δ agree on $ext(A) \cap ext(B)$.*

Proof. By definition, we have

$$C_{(A,\varepsilon)} = \bigcup_{\substack{A \in \sigma \in \Delta \\ \gamma|_{ext(A)} = \varepsilon}} \sigma^\gamma. \tag{3.1}$$

Since $\partial C_{(A,\varepsilon)}$ is the link of (A, ε) in $\pm\Delta_A$, the statement of the lemma is equivalent to

$$\partial C_{(A,\varepsilon)} = \bigcup_{\substack{\tau \in lk_{\Delta_A}(A) \\ \gamma|_{ext(A)} = \varepsilon}} \tau^\gamma = \bigcup_{\substack{B \subsetneq A \\ \delta|_{ext(A)\cap ext(B)} = \varepsilon|_{ext(A)\cap ext(B)}}} C_{(B,\delta)}. \tag{3.2}$$

Here the unions are over $\gamma : [n] \to \{\pm 1\}$ and $\delta : ext(B) \to \{\pm 1\}$, respectively, and $lk_{\Delta_A}(A) = \{\tau \in \Delta_A \mid A \notin \tau, \tau \cup \{A\} \in \Delta_A\}$ is the link of A in Δ_A.

To see the second equality in (3.2), note that if τ^γ, $\tau \in lk_{\Delta_A}(A)$, $\gamma|_{ext(A)} = \varepsilon$, appears on the left side, then $\tau^\gamma \subset C_{(B,\gamma|_{ext(B)})}$, where B is a maximal element of τ. Since $\gamma|_{ext(A)\cap ext(B)} = \varepsilon|_{ext(A)\cap ext(B)}$, the cell $C_{(B,\gamma|_{ext(B)})}$ appears on the right side.

For the opposite inclusion, suppose τ^γ is a maximal simplex of $\pm\Delta_A$ in $C_{(B,\delta)}$, where $B \subsetneq A$ and $\delta|_{ext(A)\cap ext(B)} = \varepsilon|_{ext(A)\cap ext(B)}$. Then $\gamma|_{ext(B)} = \delta$ by (3.1), and so

$$\gamma|_{ext(A)\cap ext(B)} = \delta|_{ext(A)\cap ext(B)} = \varepsilon|_{ext(A)\cap ext(B)}.$$

Since $i \in ext(A) \cap B$ implies $i \in ext(B)$ (otherwise $i \in \langle B \setminus \{i\}\rangle \subset \langle A \setminus \{i\}\rangle$), we have that the only places where $\gamma|ext(A)$ and ε might not agree are outside of B. Since $ext(\tau) \subset B$, we may, by Remark 3.1, adjust γ to γ' outside of B so that $\tau^{\gamma'} = \tau^\gamma$ and $\gamma'|_{ext(A)} = \varepsilon$. Thus τ^γ appears on the left of (3.2), establishing the equality. $\square$

We can now prove the main result of this section.

Theorem 3.5. *The cells in Σ_L form a regular CW sphere, with face poset $Q_L \setminus \hat{0}, \hat{1}$ and barycentric subdivision $\pm\Delta$.*

Proof. Since each Δ^ε is a cone on $([n], \varepsilon|_{ext([n])})$,

$$|\Sigma_L| = \bigcup_{(A,\varepsilon)\in Q_L\setminus \hat{0},\hat{1}} C_{(A,\varepsilon)} = \bigcup_{\varepsilon: ext([n])\to\{\pm 1\}} C_{([n],\varepsilon)} = |\pm\Delta|,$$

so $|\Sigma_L|$ is a sphere by Theorem 3.2.

By construction, the only inclusions $C_{(B,\delta)} \subseteq C_{(A,\varepsilon)}$ possible among cells is when $C_{(B,\delta)} \subseteq \partial C_{(A,\varepsilon)}$, so $C_{(B,\delta)} \subseteq C_{(A,\varepsilon)}$ if and only if $(B,\delta) \leq (A,\varepsilon)$ in $Q_L \setminus \hat{0}, \hat{1}$ by Lemma 3.4.

To see that $|\Sigma_L|$ is a regular CW sphere, one can assemble $|\Sigma_L|$ according to a linear extension of the poset $Q_L \setminus \hat{0}, \hat{1}$. By Lemma 3.4, all the boundary faces of any cell $C_{(A,\varepsilon)}$ will be present when it comes time to attach it.

Since the poset of inclusions among the faces of Σ_L is $Q_L \setminus \hat{0}, \hat{1}$, it will have $\pm\Delta$ as barycentric subdivision by Proposition 3.3. □

3.3. **Join-distributive lattices.** We note briefly that everything in this section works for *join-distributive lattices*, that is lattices L whose dual L^* (reverse all order relations) is meet-distributive. Here we have to reverse the roles of $\hat{0}$ and $\hat{1}$. In particular, both L and L^* have the same order complex, so we have $\Delta(L \setminus \hat{1}) = \Delta(L^* \setminus \hat{0}) = \Delta$, which gives rise to the same simplicial polytope $\pm\Delta$.

For join-distributive L, the poset $Q_L = (Q_{L^*})^*$, and so the corresponding spherical complex Σ_L is defined by defining the maximal cells to correspond to the maximal elements of $Q_L \setminus \hat{1}$ (the minimal elements of $Q_{L^*} \setminus \hat{0}$). Here, the CW sphere $\Sigma_L = (\Sigma_{L^*})^*$ is the dual to Σ_{L^*}.

Figure 4 shows the CW sphere for both the meet-distributive L from three collinear points and the corresponding join-distributive L^*. Note that both $\pm\Delta$ and Σ_L retain the full $(\mathbb{Z}/2\mathbb{Z})^n$ symmetry of the crosspolyope.

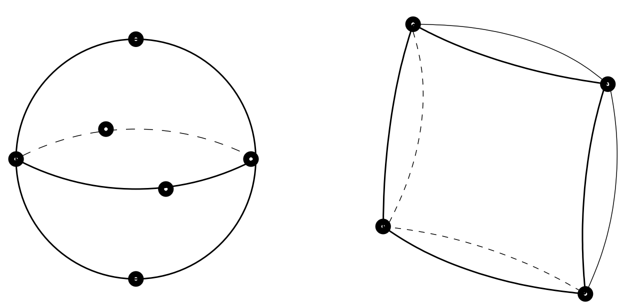

FIGURE 4. The spheres Σ_L and Σ_{L^*} for L from three collinear points.

We remark that if L is the lattice of the convex geometry on n collinear points, then one can verify that Q_{L^*} is isomorphic to the Tchebyshev poset T_n of [18]. Hetyei showed that T_n is the face poset of a regular CW sphere and its order complex subdivides a crosspolytope. In fact his proof of the latter assertion uses essentially the reflection construction discussed at the beginning of §3.1.

4. Enumerative properties of Q_L

For a graded poset P (with $\hat{0}$ and $\hat{1}$) with rank function ρ, define

$$\nu(P) = \sum_{t \in P} (-1)^{\rho(t)} \mu(\hat{0}, t). \tag{4.1}$$

If L is the intersection lattice of a real hyperplane arrangement, a well known result due to Zaslavsky [28] gives $\nu(L)$ as the number of connected components in the complement of the arrangement. He extended this to show how all the face numbers of an arrangement depend solely on the lattice of intersections. As a generalization, [9, Proposition 4.6.2] expresses the flag numbers of an arrangement, that is, the enumerators of chains of faces having prescribed rank sets, in terms of the functional ν applied to intervals in the intersection lattice.

We now show that for join-distributive L, the flag numbers of Q_L may be computed similarly from intervals in L. (Flag numbers of meet-distributive lattices can be obtained directly from this using duality.) Suppose that L consists of the closed sets of a convex geometry ordered by reverse inclusion. In analogy with the zero map on oriented matroids [9], we define the map $z : Q_L \setminus \hat{0} \to L$ by $z((A, \varepsilon)) = A$.

Proposition 4.1. *Let $c = \{A_1 < A_2 < \cdots < A_k = \hat{1}\}$ be a chain in L and $z^{-1}(c)$ denote the set of chains in Q_L that are mapped by z to c. Then*

$$|z^{-1}(c)| = \prod_{i=1}^{k-1} \nu([A_i, A_{i+1}]).$$

Proof. Given a sign function $\varepsilon_i : ext(A_i) \to \{\pm 1\}$ for some $2 \le i \le k$, there are $2^{|ext(A_{i-1}) \setminus ext(A_i)|}$ sign functions $\varepsilon_{i-1} : ext(A_{i-1}) \to \{\pm 1\}$ such that $(A_{i-1}, \varepsilon_{i-1}) < (A_i, \varepsilon_i)$ in Q_L, since the only restriction on ε_{i-1} is that it agree with ε_i on $ext(A_i) \cap ext(A_{i-1})$. Thus, starting with $\varepsilon_k = \emptyset$, there are precisely $\prod_{i=1}^{k-1} 2^{|ext(A_i) \setminus ext(A_{i+1})|}$ ways to build a sequence of sign functions $\varepsilon_k, \ldots, \varepsilon_2, \varepsilon_1$ resulting in a chain $(A_1, \varepsilon_1) < \cdots < (A_k, \varepsilon_k)$ in Q_L.

To complete the proof it suffices to show that for $1 \le i \le k$,

$$\sum_{A_i \le B \le A_{i+1}} (-1)^{\rho(A_i,B)} \mu(A_i,B) = 2^{|ext(A_i) \setminus ext(A_{i+1})|}. \tag{4.2}$$

The Möbius function of a join-distributive lattice satisfies

$$(-1)^{\rho(A_i,B)} \mu(A_i,B) = \begin{cases} 1 & \text{if } [A_i,B] \text{ is a Boolean lattice,} \\ 0 & \text{otherwise.} \end{cases}$$

(This follows, for example, from [13, Theorems 4.2, 4.3].) By definition of join-distributivity, for $B \in [A_i, A_{i+1}]$ the interval $[A_i, B]$ is a Boolean lattice precisely when B is less than or equal to the join of atoms of $[A_i, A_{i+1}]$, which are those $A_i \setminus a$ such that $a \in ext(A_i) \setminus ext(A_{i+1})$. Hence the left side of (4.2) reduces to a sum of the form $\sum_B 1$ with B ranging over a Boolean lattice of rank $|ext(A_i) \setminus ext(A_{i+1})|$. □

The complete enumerative information on chains in a graded poset P is carried by the formal power series

$$F_P := \sum_{\substack{\hat{0}=t_0<t_1<\cdots<t_k=\hat{1} \\ 0<i_1<\cdots<i_k}} x_{i_1}^{\rho(t_0,t_1)} x_{i_2}^{\rho(t_1,t_2)} \cdots x_{i_k}^{\rho(t_{k-1},t_k)},$$

where $\rho(s,t) = \rho(t) - \rho(s)$. As P ranges over the family of graded posets, the F_P span the (Hopf) algebra of quasisymmetric functions, denoted $\mathcal{Q}$. The definition of F_P is due to Ehrenborg [16]. See [24, §7.19] for further background on quasisymmetric functions.

In the context of combinatorial Hopf algebras [1], the functional ν can be seen as the pullback of a certain "odd character" $\nu_{\mathcal{Q}}$ to the Hopf algebra of graded posets along the map $P \mapsto F_P$; that is, $\nu(P) = \nu_{\mathcal{Q}}(F_P)$. By the general theory there is an induced Hopf algebra map $\vartheta : \mathcal{Q} \to \mathcal{Q}$ satisfying

$$\vartheta(F_P) = \sum_{\substack{\hat{0}=t_0<t_1<\cdots<t_k=\hat{1} \\ 0<i_1<\cdots<i_k}} \nu([t_0,t_1]) \cdots \nu([t_{k-1},t_k]) x_{i_1}^{\rho(t_0,t_1)} \cdots x_{i_k}^{\rho(t_{k-1},t_k)}.$$

In fact ϑ is precisely the map introduced by Stembridge [25] to relate the quasisymmetric weight enumerator for P-partitions of a labeled poset to the enriched quasisymmetric weight enumerator of that poset. See [1, Examples 2.2, 4.4, 4.9].

The main result of this section is an extension of [19, Theorem 5.15]:

Theorem 4.2. *For a join-distributive lattice L, we have*

$$2F_{Q_L} = \vartheta(F_{L \cup \hat{0}}),$$

where $\hat{0}$ denotes a new minimum element adjoined to L.

Proof. This is essentially the argument used to prove [5, Theorem 3.1]. Extend z to a map $z : Q_L \to L \cup \hat{0}$ by requiring $z(\hat{0}) = \hat{0}$. By Proposition 4.1,

$$\begin{aligned}
F_{Q_L} &= \sum_{\substack{c=\{\hat{0}=A_0<\cdots<A_k=\hat{1}\}\subset L\cup\hat{0}\\ i_1<\cdots<i_k}} |z^{-1}(c)| x_{i_1}^{\rho(A_0,A_1)} \cdots x_{i_k}^{\rho(A_{k-1},A_k)} \\
&= \sum_{\substack{\hat{0}=A_0<\cdots<A_k=\hat{1}\\ i_1<\cdots<i_k}} \nu([A_1,A_2]) \cdots \nu([A_{k-1},A_k]) x_{i_1}^{\rho(A_0,A_1)} \cdots x_{i_k}^{\rho(A_{k-1},A_k)} \\
&= \frac{1}{2} \sum_{\substack{\hat{0}=A_0<\cdots<A_k=\hat{1}\\ i_1<\cdots<i_k}} \nu([A_0,A_1]) \cdots \nu([A_{k-1},A_k]) x_{i_1}^{\rho(A_0,A_1)} \cdots x_{i_k}^{\rho(A_{k-1},A_k)} \\
&= \frac{1}{2}\vartheta(F_{L\cup\hat{0}}).
\end{aligned}$$

The third equality holds because $\hat{0}$ is covered by only one element in $L \cup \hat{0}$, implying that $\mu(\hat{0}, A_1)$ vanishes if $\rho(A_1) > 1$; hence $\nu([\hat{0}, A_1]) = \mu(\hat{0}, \hat{0}) - \mu(\hat{0}, [n]) = 2$. ☐

Analogously, if Z is the face lattice of the zonotope associated to an arrangement $\mathcal{A}$ and L is the intersection lattice of $\mathcal{A}$, then

$$2F_Z = \vartheta(F_{L\cup\hat{0}})$$

[5] [3, Proposition 3.5]. It is easy to see that a join-distributive lattice L must be semimodular, as are geometric lattices. One is led to speculate whether this relationship holds for *all* semimodular lattices, namely, whether for any semimodular lattice L, there exists a regular CW sphere Σ_L with face poset Q_L (with $\hat{0}$, $\hat{1}$ adjoined) such that

$$2F_{Q_L} = \vartheta(F_{L\cup\hat{0}}).$$

The role played by convex closures in this work might be played instead by *interval greedoids* (see [11, Theorem 8.8.7]). Note that this would imply the existence of spheres Σ_L for geometric lattices that are not necessarily orientable. In nonorientable case, one might also ask for the relationship of $\vartheta(F_{L\cup\hat{0}})$ to the (dual) face counts of the homotopy-sphere arrangements of Swartz [27]. Simple examples suggest the former might provide lower bounds for the latter. In the orientable case, these bounds are clearly achieved by the results of [3, 5]. One could speculate further that achieving the bounds implies orientability.

We conclude with some remarks on the relation between enumeration of faces in $\Delta(Q_L \setminus \hat{0}, \hat{1})$ and enumeration of enriched P-partitions.

For a $(d-1)$-dimensional simplicial complex Δ having f_{i-1} faces of cardinality i $(0 \leq i \leq d)$, the *h-polynomial* of Δ, denoted $h^{\Delta}(t)$, may be defined by

$$h^{\Delta}(t) = \sum_{i=0}^{d} f_{i-1}(1-t)^{d-i}t^{i}$$

(see, for example, [4]).

The *enriched order polynomial* of a naturally labeled poset P, denoted $\overline{\Omega}_P(t)$, is determined by the property that $\overline{\Omega}_P(m)$ is the number of order-preserving maps $\sigma : P \to [2m]$ such that $\sigma(x) = \sigma(y)$ implies $\sigma(x)$ is even. Enriched order polynomials are fundamental in the study of enriched P-partitions, which bear an analogous relation to Schur's Q-functions as ordinary P-partitions do to ordinary Schur functions [25].

As a direct consequence of [19, Corollary 5.3] [23, Chapter 3, Exercise 67b], we have

Proposition 4.3. *Suppose that P is a naturally labeled poset of size n with minimum element 0_P and $L = J((P \setminus 0_P)^*)$ is the distributive lattice of lower order ideals of $(P \setminus 0_P)^*$. Then*

$$\sum_{m \geq 0} \overline{\Omega}_P(m)t^m = 2t \cdot \frac{h^{\Delta(Q_L \setminus \hat{0},\hat{1})}(t)}{(1-t)^{n+1}}.$$

Stembridge found examples of a naturally labeled poset P (with minimum element) such that the numerator of $\sum_{m \geq 0} \overline{\Omega}_P(m)t^m$ has non-real roots [26]. From such a poset one can construct via our results a flag simplicial complex (meaning every minimal non-face has size two), namely $\Delta(Q_L \setminus \hat{0}, \hat{1})$, that barycentrically subdivides a regular CW sphere and whose h-polynomial has non-real roots. In fact, this simplicial sphere will be a simplicial polytope. Until recently it had been conjectured that the h-polynomial of a flag simplicial triangulation of a sphere should have only real roots. The first counterexamples were given by Gal [17].

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