One-and-Two Target Distributed Optimal Control of a Parabolic System

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Abstract

In this paper models for two-target optimal controls of a linear diffusion equation are considered. That is, we seek to control the solution of the given PDE in order to guide it to a specified target within a given error at some fixed finite time. At the same time it is desired to keep the variation of the solution from a second target as small as possible. Thus the problem is essentially a multi-objective optimization problem but with a clearly defined primary goal. Two different models for this problem are studied and the primal and dual problems associated with these models are formulated with a goal of finding algorithmic approaches to solving the two-target problem. An analysis of two versions of the one-target model is provided as a basis for the study of the two-target model.

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1 Introduction

In this paper we Consider the partial differential equation:

$$\begin{aligned} \frac{dy}{dt} + Ay &= V(x,t) \qquad (x,t) \in \Omega \times (0,T) \\ y(x,0) &= 0 \qquad x \in \Omega \\ y(x,t) &= 0 \qquad (x,t) \in \Gamma \times (0,T), \end{aligned} \tag{1}$$

where A is a uniformly elliptic differential and self-adjoint operator (e.g., the Laplacian), Ω is a bounded open subset of R^n, $T > 0$ is finite, Γ is the boundary of Ω. $V(x,t)$ is the control function that is used to control the state vector $y(x,t)$. Of primary interest is the case where the control is applied in two sets having fixed centers, i.e., $V(x,t) = v_1(t))\, \chi_{O_1} + v_2(t)\, \chi_{O_2}$, where χ_{O_j} is the characteristic function of an open set centered $a_j \in (0,L), j = 1,2$ points in the interior of Ω. The object is to choose $v_1(t)$ and $v_2(t)$ so that the state of the system at time T, $y(x,T)$ is close to two different $L^2(\Omega)$ targets y_{T1} and y_{T2} with the added assumption that y_{T1} is the primary target, in the sense that the deviation from it should be as small as possible. The goal of this study is to compare several different approaches to modeling this problem and to analyze the resulting primal and dual optimization problems. In a subsequent paper, some numerical studies relating these problems will be investigated.

Although the examination of the two target case is the goal we provides a good introduction to the theoretical approach that we take in analyzing the situation in which there are two targets.

2 The One-Target Problem With Distributed Control

In this section the primal and dual problems when there is one target are discussed. The discussion follows the development in [GL94]. When $V(x,t) = v(x,t)\chi_O$ where χ_O is the characteristic function for a fixed proper open subset O, of Ω and $v(x,t) \in L^2(O \times (0,T))$ the problem has the form:

$$(P1) \qquad \min \quad \frac{1}{2} \int_{O \times (0,T)} v(x,t)^2 \chi_O \; dx\, dt$$

subject to

$$
\begin{aligned}
y_t + Ay &= v(x,t)\chi_O \\
y(x,0) &= 0 \qquad x \in \Omega \\
y(x,t) &= 0 \qquad (x,t) \in \Gamma \times (0,T) \\
\int_\Omega (y(x,T) - y_T(x))^2 dx &\leq \beta^2
\end{aligned}
$$

It is well known that the equations and inequality have solutions for any $\beta > 0$, i.e., the problem is approximately controllable. With the quadratic convex objective function the optimal control problem has a unique L^2 solution.

A second formulation of the one-target problem is:

$$
(P2) \qquad \min \quad \frac{1}{2}\int_{O\times(0,T)} v(x,t)^2 \chi_O dxdt + \frac{\alpha}{2}\int_\Omega (y(x,T) - y_T(x))^2 dx
$$

subject to

$$
\begin{aligned}
y_t + Ay &= v(x,t)\chi_O \\
y(x,0) &= 0 \qquad x \in \Omega \\
y(x,t) &= 0 \qquad (x,t) \in \Gamma \times (0,T)
\end{aligned}
$$

Presumably for α sufficiently large, a solution of this problem will satisfy the constraint

$$
\int_\Omega (y(x,T) - y_T(x))^2 dx \leq \beta^2
$$

although it needs not be optimal for the problem $(P1)$. Generally speaking, this second formulation is easier to solve than the first since the inequality constraints make the problem more difficult. Therefore, it makes sense to solve the latter problem for α large enough that the above constraint is satisfied. The question is: How large the α have to be? An approach to answering this question is outlined below.

The associated adjoint system for the system (1) is defined by

$$
\begin{aligned}
p_t + Ap &= 0 \qquad (x,t) \in \Omega \times (0,T) \qquad (2) \\
y(x,T) &= w(x) \qquad x \in \Omega \\
y(x,t) &= 0 \qquad (x,t) \in \Gamma \times (0,T),
\end{aligned}
$$

for $w \in L^2(\Omega)$. The pair of the systems (1) and (2) lead directly to the solution of $(P2)$ in the following way.

Theorem 1 *If $w(x)$ is chosen as $-\alpha(y(x,T) - y_T(x))$ and $v(x,t)$ is set to be $p(x,t)$, then the simultaneous solution of the strongly coupled systems (1) and (2) yields the optimal state vector $y(x,t)$, for $(P2)$.*

This result is obtained from variational equation for $(P2)$; see [GL94].

Since there is no similar result for $(P1)$, the relationship between the problems $(P1)$ and $(P2)$, i.e., between β and α, is not obvious. However, some insight can be gained by considering their associated dual problems. The conjugate duality theory of [Roc74], [ET74], et al, can be used to derive these problems. For its application to these cases let V and W be two Hilbert spaces and let L be a bounded linear transformation from V to W. If $f_1 : V \to R^e$ and $f_2 : W \to R^e$ are proper convex functions (R^e denotes the extended reals) then the duality theory gives the equivalence of two optimization problems:

$$\inf_{v \in V} \{f_1(v) + f_2(L(v))\} = - \inf_{w \in W} \{f_1^*(L^*(w)) + f_2^*(-w)\}$$

where f_j^* are conjugate functions of f_j, $j = 1, , 2$. That is,

$$f_1^*(v) = \sup_{v^* \in V} \{< v, v^* >_V - f_1(v^*)\}$$

and

$$f_2^*(w) = \sup_{w^* \in W} \{< w, w^* >_W - f_2(w^*)\}$$

where $< \cdot, \cdot >$ is the inner product in the appropriate Hilbert space. L^* is the adjoint of L, i.e., L^* is a linear transformation from W to V such that

$$< v, L^*(w) >_V = < L(v), w >_W$$

for every $v \in V$ and $w \in W$.

For $(P1)$ and $(P2)$ we choose as our Hilbert spaces: $V = L^2(O \times (0,t)$ and $W = L^2(\Omega)$ and define the linear mapping $L : V \to W$ by

$$L(v) = y(x,T)$$

obtained from the corresponding solution of the system (1).

Proposition 1 *Define* $L^* : W \to V$ *by* $L^*(w) = p(x,t)\chi_O$ *where* $p(x,t)$ *is obtained from the solution of the system (2) for the given* w. *Then* L^* *is the adjoint of* L.

Proof: Multiply the PDE in (1) by $p(x,t)$ and integrate over $\Omega \times (0,T)$. Using integration by parts and Green's theorem yields

$$0 = -\int_{\Omega\times(0,T)} (p_t + Ap)y \; dx\, dt + \int_{\Omega} y(x,T)p(x,T) \; dx = \int_{\Omega\times(0,T)} v(x,t)p(x,t)\chi_O \; dxdt$$

Therefore, using (2) gives

$$\int_{\Omega} y(x,T)w(x)dx = \int_{\Omega\times(0,T)} v(x,t)p(x,t)\chi_O \; dx\, dt$$

or

$$< L(v), w >_{L^2(\Omega)} = < v, L^*(w) >_{O\times(0,T)}$$

End of Proof

Now define the functions

$$\begin{aligned} f_1(v) &= \frac{1}{2}\int_{O\times(0,T)} v(x,t)^2\chi_O dxdt \\ f_2(w) &= \frac{\alpha}{2}\int_{\Omega} (w - y_T(x))^2 dx \\ f_3(w) &= \begin{cases} 0 & \text{if } \|w(x) - y_T\|_{L^2(\Omega)} \leq \beta \\ +\infty & \text{otherwise} \end{cases} \end{aligned}$$

Straightforward calculation show that the corresponding conjugate functions are

$$\begin{aligned} f_1^*(v) &= \frac{1}{2}\int_{O\times(0,T)} v(x,t)^2\chi_O dxdt \\ f_2^*(w) &= < w, y_T >_{L^2(\Omega)} + \frac{1}{2\alpha}\|w\|^2_{L^2(\Omega)} \\ f_3^*(w) &= < w, y_T >_{L^2(\Omega)} + \beta\|w\|_{L^2(\Omega)} \end{aligned}$$

Thus the dual problem for $(P1)$ is

$$-\inf_{w\in L^2(\Omega)} \{\frac{1}{2}\int_{O\times(0,T)} p(x,t)^2\chi_O dxdt - < w, y_T >_{L^2(\Omega)} + \beta\|w\|_{L^2(\Omega)}\}$$

and the dual problem for $(P2)$ is

$$-\inf_{w\in L^2(\Omega)} \{\frac{1}{2}\int_{O\times(0,T)} p(x,t)^2\chi_O dxdt - < w, y_T >_{L^2(\Omega)} + \frac{1}{2\alpha}\|w\|^2_{L^2(\Omega)}\}$$

where $p(x,t)$ is obtained by solving (2) for the given w.

For $w \in L^2(\Omega)$ let $y(x,t)$ and $p(x,t)$ be the solutions of (1) and (2) with $v(x,t) = p(x,t)\chi_O$ and define the linear operator $\Lambda : L^2(\Omega) \to L^2(\Omega)$ by

$$\Lambda(w) = y(x,T)$$

Note that $\Lambda = L(L^*(w))$ where L and L^* are the linear operators given above.

Proposition 2 *Λ is a self-adjoint and nonnegative definite linear transformation satisfying*

$$< \Lambda(w), w >_{L^2(\Omega)} = \int_{O\times(0,T)} p(x,t)^2 \chi_O \; dx\, dt$$

Proof: For w_1 and w_2 in $L^2(\Omega)$ let p_1 and p_2 be the solutions to (2) and set $y_1 = \Lambda(w_1)$ and $y_2 = \Lambda(w_2)$. The from the properties of L and L^*.

$$\begin{aligned} < \Lambda(w_1), w_2 >_{L^2(\Omega)} &= < L(L^*(w_1)), w_2 >_{L^2(\Omega)} \\ &= < L^*(w_1), L^*(w_2) >_{O\times(0,T)} \\ &= < p_1(x,t)\chi_O, p_2(x,t)\chi_O >_{O\times(0,T)} \end{aligned}$$

The property of Λ follows immediately. *End of Proof*

The dual problems for $(P1)$ and $(P2)$ can now be written as

(1.1) $$-\inf_{w\in L^2(\Omega)} \{< \Lambda(w_1), w_2 >_{L^2(\Omega)} - < w, y_T >_{L^2(\Omega)} + \beta \|w\|_{L^2(\Omega)}\}$$

and

(1.2) $$-\inf_{w\in L^2(\Omega)} \{< \Lambda(w_1), w_2 >_{L^2(\Omega)} - < w, y_T >_{L^2(\Omega)} + \frac{1}{2\alpha} \|w\|^2_{L^2(\Omega)}\}$$

respectively.

Since the inequalities (1.3) are just the variational inequalities for the problem (1.1), we have:

Theorem 2 *Solving problem $(P1)$ is equivalent to solving the problem: find $w^* \in L^2(\Omega)$ such that for every $w \in L^2(\Omega)$*

(1.3) $$< \Lambda(w^*), w - w^* >_{L^2(\Omega)} + \beta(\|w\|_{L^2(\Omega)} - \|w^*\|_{L^2(\Omega)}) \geq < w - w^*, y_T >_{L^2(\Omega)}$$

in that if w^* satisfies this equation then the corresponding $p^*(x,t)$ given by (2) yields the optimal control for $(P1)$ and the resulting $y^*(x,t)$ given by (1) is the optimal state variable.

In addition, since the solution of (1.4) minimizes the positive definite quadratic form given in (1.2) we have:

Theorem 3 *Solving problem $(P2)$ is equivalent to solving the linear system*

$$(\Lambda + \frac{1}{\alpha} I)w = y_T \tag{1.4}$$

in that if w^* satisfies this equation then the corresponding $p^*(x,t)$ given by (2) yields the optimal control for $(P2)$ and the resulting $y^*(x,t)$ given by (1) is the optimal state variable.

Note that although computing $\Lambda(w)$ involves solving (1) and (2), it does not require them to be solved simultaneously since (2) can be solved independently of (1). Since the operator in (1.4) is positive definite, one approach to solving $(P2)$ might be to solve a discretized version of this system by means of conjugate gradient type of algorithm.

To establish a relationship between two dual problems, suppose that w^* is the solution to the dual of $(P1)$. Then, as suggested in [GL94] substituting $w = 0$ and $w = 2w^*$ into (1.3) yields inequalities

$$- < \Lambda(w^*), w^* >_{L^2(\Omega)} - \beta \|w^*\|_{L^2(\Omega)} \geq - < w^*, y_T >_{L^2(\Omega)}$$

and

$$< \Lambda(w^*), w^* >_{L^2(\Omega)} + \beta \|w^*\|_{L^2(\Omega)} \geq < w^*, y_T >_{L^2(\Omega)}$$

which together show that the solution to the dual of $(P1)$ satisfies

$$< \Lambda(w^*), w^* >_{L^2(\Omega)} + \beta \|w^*\|_{L^2(\Omega)} = < w^*, y_T >_{L^2(\Omega)} .$$

Now let $\hat{w}$ be the solution to the dual of $(P2)$, i.e., it satisfies (1.4). Then multiplication of (1.4) by $\hat{w}$ and integration over Ω gives

$$< \Lambda(\hat{w}), \hat{w} >_{L^2(\Omega)} + \frac{1}{\alpha} \|\hat{w}\|^2_{L^2(\Omega)} = < \hat{w}, y_T >_{L^2(\Omega)}$$

Thus if the two solutions were to the same, $\hat{w} = w^*$, then the following would have to be satisfied

$$\alpha = \frac{\|w^*\|_{L^2(\Omega)}}{\beta} \tag{1.5}$$

.

This suggests the following algorithm for finding an approximate solution to the problem $(P1)$ using the solution to $(P2)$.

1. Given $\alpha^0 > 0$, set $k = 0$.

2. Solve the dual problem (1.4) with $\alpha = \alpha^k$ to obtain a solution w^k.

3. If $\|\Lambda(w^k) - y_T\|_{L^2(\Omega)} > \beta$.

   (a) Set $\alpha^{k+1} = \frac{1}{\beta}\| \, w^k\|_{L^2(\Omega)}$.

   (b) Set $k = k + 1$ and return to step 2.

4. Stop with approximate solution w^k to problem $(P1)$.

2.1 The Two-Target Problem With Distributed Control

The formulation of the two-target problem in which it is desired to minimize the deviation from the target y_{T_2} subject to the condition that the deviation from the target y_{T_1} be less than a prescribed value β is:

$$(P3) \qquad \min \quad \frac{1}{2}\int_{O\times(0,T)} (v_1^2(x,t)\chi_{O_1} + v_2^2(x,t)\chi_{O_2})dxdt + \frac{\alpha_2}{2}\int_\Omega (y(x,T) - y_{T_2}(x))^2 dx$$

subject to

$$\begin{aligned}
y_t - Ay &= v_1(x,t)\chi_{O_1} + v_2(x,t)\chi_{O_2} \\
y(x,0) &= 0 \qquad x \in \Omega \\
y(x,t) &= 0 \qquad (x,t) \in \Gamma \times (0,T) \\
\int_\Omega (y(x,T) - y_{T_1}(x))^2 dx &\leq \beta^2
\end{aligned}$$

O_1 and O_2 are proper open sets of Ω. Problem $(P3)$ has the same form as problem $(P3)$, differing only in that the objective function is more complicated, and the proof of the approximate controllability is essentially the same. The analysis of the primal and dual problems for $(P3)$ are also similar to the that of $(P1)$.

The appropriate Hilbert spaces for the analysis of $(P3)$ are

$$V = L^2(O_1 \times (0,T)) \times L^2(O_2 \times (0,T)) \tag{1.6}$$

and $W = L^2(\Omega)$. The linear mapping is given by $L : V \to W$

$$L(v_1, v_2) = y(x, T) \tag{1.7}$$

where y is the solution of the system (1) (with $V(x,t) = v_1(x,t)\chi_{O_1} + v_2(x,t)\chi_{O_2})$. The adjoint operator is $L^* : W \to V$ given by $L^*(w) = (p(x,t)\chi_{O_1} + p(x,t)\chi_{O_2}$ where $p(x,t)$ is the solution of the adjoint system (2).

Defining the functions

$$f_1(v_1, v_2) = \frac{1}{2}\int_{\Omega\times(0,T)} (v_1^2(x,t)\chi_{O_1} + v_2^2(x,t)\chi_{O_2})dxdt$$

$$f_2(w) = \begin{cases} \frac{\alpha_2}{2}\|w - y_{T_2}\|^2_{L^2(\Omega)} & \text{if } \|w - y_{T_1}\|_{L^2(\Omega)} \le \beta^2 \\ +\infty & \text{otherwise} \end{cases}$$

yields conjugate functions

$$f_1^*(v_1, v_2) = \frac{1}{2}\int_{\Omega\times(0,T)} (v_1^2(x,t)\chi_{O_1} + v_2^2(x,t)\chi_{O_2})\; dx\, dt$$

and

$$f_2^*(w) = \begin{cases} < w, y_{T_2} >_{L^2(\Omega)} + \frac{1}{2\alpha_2}\|w\|_{L^2(\Omega)} & \text{if } \|w - \alpha_2 \hat{y}_T\}\| < \beta\alpha_2 \\ \beta\|w - \alpha_2\hat{y}_T\|_{L^2(\Omega)} - \frac{\alpha_2}{2}\beta^2 + < w, y_{T_1} > - \frac{\alpha_2}{2}\|\hat{y}_T\|^2 & \text{otherwise} \end{cases}$$

where $\hat{y}_T = y_{T_1} - y_{T_2}$.

Using the Rockafellar conjugate duality result gives the dual problem or (P3)

$$-\inf_{w\in L^2(\Omega)} \{\frac{1}{2}\int_{\Omega\times(0,T)} p(x,t)^2\chi_{O_1} + p(x,t)^2\chi_{O_2}\; dx\, dt + f_2^*(-w)\}$$

As in the one-target case, the operator $\Lambda_2 : L^2(\Omega) \to L^2(\Omega)$ is defined by

$$\Lambda_2(w) = y(x, T) \tag{1.8}$$

where $y(x,T)$ is the solution of (1) when $v_1(x,t) = p(x,t)\chi_{O_1}$ and $v_2(x,t) = p(x,t)\chi_{O_2}$ and $p(x,t)$ is the solution of (2). Using the same arguments as in the proof for (P1) the following form for dual problem of $(P3)$ can be derived.

Theorem 4 *Solving problem $(P3)$ is equivalent to solving the problem: find $w^* \in L^2(\Omega)$ such that for every $w \in L^2(\Omega)$*

$$(1.9) \qquad < \Lambda_2(w^*), w - w^* >_{L^2(\Omega)} + f_2^*(w) - f_2^*(w^*) \geq 0$$

Both the primal and dual problems of $(P3)$ are difficult to solve. Consequently, alternative formulations that require only (linear) equality constraints for the primal problem or only the solution of a linear system for the dual problem would be preferable. Two alternative formulations are suggested here.

The first alternative is the analogue of problem $(P2)$ in the previous section. This problem has the form

$$(P4) \quad \min \quad \frac{1}{2}\int_{O\times(0,T)} (v_1^2(x,t)\chi_{O_1} + v_2^2(x,t)\chi_{O_2})\, dx\, dt + \frac{\alpha_1}{2}\|(y - y_{T_1}\|^2 + \frac{\alpha_2}{2}\|(y - y_{T_2}\|^2$$

subject to

$$\begin{aligned} y_t - Ay &= v_1(x,t)\chi_{O_1} + v_2(x,t)\chi_{O_2} \\ y(x,0) &= 0 \qquad x \in \Omega \\ y(x,t) &= 0 \qquad (x,t) \in \Gamma \times (0,T) \end{aligned}$$

In this model the emphasis on the deviations from the two targets is determined by the relative sizes of the parameters α_1 and α_2.

The direct solution of $(P4)$ can be found in a manner similar to that of $(P2)$. In particular, one obtains:

Theorem 5 *If $w(x)$ is chosen as*

$$w = -(\alpha_1 + \alpha_2)y(x,T) + \alpha_1 y_{T_1} + \alpha_2 y_{T_2}$$

and $v_i(x,t)$ are chosen as $p(x,t)\chi_{O_i}$, $i = 1,2$, then the simultaneous solutions of (1) and (2) yields the optimal state vector, $y(x,t)$ for $(P4)$

The dual formulation for $(P4)$ can also be obtained by following the steps for the dual problem $(P2)$ with the functions f_2 and f_2^* slightly modified. Specifically, define the Hilbert spaces V and W and the linear mappings L and Λ_2 as in (1.6), (1.7), and (1.8). Also let $f_1(v_1, v_2)$ and hence f_1^* be defined as for $(P3)$. Defining

$$\hat{f}_2(w) = \frac{\alpha_1}{2}\|w - y_{T_1}\|^2_{L^2(\Omega)} + \frac{\alpha_2}{2}\|w - y_{T_2}\|^2_{L^2(\Omega)}$$

gives the conjugate function

$$\hat{f}_2^*(w) = < w, \frac{\alpha_1 y_{T_1} + \alpha_2 y_{T_2}}{\alpha_1 + \alpha_2} >_{L^2(\Omega)} + \frac{1}{2(\alpha_1 + \alpha_2)}\|w\|^2_{L^2(\Omega)} - \frac{\alpha_1\alpha_2}{2(\alpha_1 + \alpha_2)}\|y_{T_1} - y_{T_2}\|^2_{L^2(\Omega)}$$

Using the duality theory as before the following duality result for $(P4)$ is obtained.

Theorem 6 *Solving the problem* $(P4)$ *is equivalent to solving the linear system*

$$(\Lambda_2 + \frac{1}{\alpha_1 + \alpha_2} I)(w) = (\frac{1}{\alpha_1 + \alpha_2})(\alpha_1 y_{T_1} + \alpha_2 y_{T_2})$$

The task now is, for a given β, to determine constants α_1 and α_2 (in an iterative scheme) which will force the constraint

$$\int_\Omega (y(x,T) - y_{T_1}(x))^2 dx \leq \beta^2$$

to be satisfied as in the one-target case. This problem is still being investigated.

We note that there are other models for this two-target problem; in particular, the hierarchical models introduced in [BKT03]. We will investigate these models in a later paper.

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