Efficient Mean Estimation in Lognormal Linear Models

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Summary. Lognormal linear models are widely used in applications, and many times it is of interest to predict the response variable at the original scale for a new set of covariate values. In this paper we consider the problem of efficient estimation of the conditional mean of the response variable at the original scale for lognormal linear models. Several existing estimators are reviewed, including the maximum likelihood (ML) estimator, the restricted ML (REML) estimator, the uniformly minimum variance unbiased (UMVU) estimator, and a bias-corrected REML estimator. We propose two estimators that minimize the asymptotic mean squared error (MSE) and the asymptotic bias respectively, as well as a bootstrap method to obtain their corresponding confidence intervals. Comparisons of the estimators using simulation studies demonstrate that even for a small sample size and a moderately large σ^2,

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our estimators are at least as good as, and many times better than the best existing estimators, and the bootstrap procedure yields confidence intervals with smaller coverage bias. Both the new estimators and the bootstrap procedure are very easy to implement. A real data example of predicting the sediment discharge is used to illustrate the methodology.

KEY WORDS: maximum likelihood; parametric bootstrap; mean squared error; uniformly minimum variance unbiased.

1. Introduction

The prevalence of lognormality has been reported in a wide range of applications from mining (Marcotte and Groleau, 1997), estimating insurance reserves (Doray, 1996), water quality control (Gilliom and Helsel, 1986), to monitoring air pollution concentrations (Holland et al., 2000) and estimating the sediment discharge (Cohn, 1995), to name just a few. Lognormal linear models are often used in these applications, in which regression models are fitted to logarithmic transformed response variables. To be more specific, let $Z = (Z_1, \ldots, Z_n)^T$ be the lognormal response vector, and $x_i = (1, x_{i1}, \ldots, x_{ip})^T$ the covariate vector for observation i. A lognormal linear model assumes that

$$Y = \log(Z) = X\beta + \epsilon, \tag{1}$$

where $X = (x_1, \ldots, x_n)^T$, $\beta = (\beta_0, \beta_1, \ldots, \beta_p)^T$, and $\epsilon = (\epsilon_1, \ldots, \epsilon_n)^T$ with $\epsilon_i \stackrel{i.i.d.}{\sim} N(0, \sigma^2)$. In many cases, one is interested in predicting the response variable for a new set of covariate value x_0 at the original scale, i.e., predicting $Z_0 = \exp(x_0^T\beta + \epsilon_0)$, where ϵ_0 is the corresponding normal error with mean zero and variance σ^2. It is well known that if β and σ^2 are known, the best predictor under the squared error loss is the conditional expectation,

$$\mu(x_0) = \mathrm{E}(Z_0|x_0) = \exp\left(x_0^T\beta + \frac{1}{2}\sigma^2\right). \tag{2}$$

Note that $\mu(x_0)$ is a function of the unknown parameters β and σ^2, which have to be estimated in practice.

In this paper we consider the problem of efficient estimation of $\mu(x_0)$, the conditional expectation of Z at x_0. A number of authors have considered this problem in the past. Finney (1941) developed the uniformly minimum variance unbiased (UMVU) estimator of parameters of the lognormal distribution and Heien (1968) extended Finney's method to simple lognormal regression models. Bradu and Mundlak (1970) derived the UMVU estimator and its variance for general lognormal regression models. Maximum likelihood (ML) and restricted maximum likelihood (REML) methods have also been used in practice. A general discussion can be found in, for example, Ch. 6 of Lawless (1982). More recently, El-shaarawi and Viveros (1997) proposed a bias corrected REML estimator, (the EV estimator). We will review all these estimators in Section 2, and compare their performance in Section 4.

A common measure of the quality of an estimator in the statistical literature is the Mean Squared Error (MSE) defined as

$$\mathrm{MSE}\left[\widehat{\mu}(x_0)\right] = \mathrm{E}\left[\widehat{\mu}(x_0) - \mu(x_0)\right]^2 = \mathrm{Var}\left[\widehat{\mu}(x_0)\right] + \mathrm{Bias}^2\left[\widehat{\mu}(x_0)\right], \tag{3}$$

where $\mathrm{Bias}\left[\widehat{\mu}(x_0)\right] = \mathrm{E}\left[\widehat{\mu}(x_0)\right] - \mu(x_0)$ is the bias of the estimator $\widehat{\mu}(x_0)$. In terms of MSE, the UMVU estimator is the best estimator among all unbiased estimators of $\mu(x_0)$. However, the UMVU estimator can only be expressed as the sum of an infinite series of Hypergeometric functions (Seaborn, 1991), which is not very convenient to use for practitioners. Furthermore, one can find better estimators in terms of MSE if one can tolerate a small bias. The EV estimator, for example, has a small bias, while its MSE is smaller than the UMVU estimator in many cases (see Section 4 for more details).

In this paper, we investigate a class of estimators obtained by plugging in the ML estimator for β and a RSS-based degree of freedom (d.f.) adjusted estimator for σ^2 in (2). Both the ML and REML estimators belong to this class of estimators. We propose two new estimators of $\mu(x_0)$ from this class, the minimum MSE (MM) estimator and the minimum bias (MB) estimator, which minimizes the asymptotic MSE and bias respectively. In practice, one may use either estimator depending on the trade off between bias and MSE that one wants to achieve for a particular application. Our results also show that the d.f. for estimating σ^2 should be between the d.f.s for the MM and MB estimators, no matter what kind of trade off one wants to achieve between bias and MSE, and the REML estimator without bias correction should never be used for estimating $\mu(x_0)$ under any circumstances.

A direct comparison with all the estimators mentioned above indicates that our estimators have superior performance in terms of MSE or bias, and for small sample sizes, the improvement is rather substantial. We also give the corresponding confidence intervals for our estimators using a parametric bootstrap method, which are shown to have nice coverage properties.

Section 2 reviews the existing methods for estimation of lognormal linear models. The estimators as well as their bias and MSE are given there. In Section 3 we derive the MM and MB estimators with their bias and MSE. A parametric bootstrap procedure is also presented to derive the corresponding confidence intervals. The MSEs of the MM and MB estimators are compared with the existing estimators in Section 4 for a range of sample sizes and σ^2 values that cover most practical situations, and empirical coverage properties of the bootstrap confidence intervals are also investigated. One real application to a Sediment discharge data is shown in Section 5 to illustrate the practical comparison of the different estimators. All the technical details are relegated to the Appendix.

2. Existing estimators for lognormal regression models

In this section we derive the bias and MSE of the existing estimators under the model assumption (1). We first give several well-known results about the distributions for the ordinary least squares (OLS) estimators for β and the residual sum of squares (RSS) in the following two propositions.

PROPOSITION 1. *The OLS estimator for β is*

$$\widehat{\beta} = \left(X^T X\right)^{-1} X^T Y \sim N\left(\beta, \sigma^2 \left(X^T X\right)^{-1}\right).$$

As a result, $x_0^T \widehat{\beta} \sim N\left(x_0^T \beta, \sigma^2 v_0\right)$ where $v_0 = x_0^T \left(X^T X\right)^{-1} x_0$. In addition, if $X^T X = O(n)$ and x_0 is bounded, then $v_0 = O(1/n)$.

PROPOSITION 2. *Let $m = n - (p+1)$. The residual sum of squares is*

$$RSS = Y^T \left[I - X\left(X^T X\right)^{-1} X^T\right] Y \sim \sigma^2 \chi^2_m.$$

Thus, the moment generating functions (MGF) of RSS is

$$E\left[\exp(cRSS)\right] = \left(1 - 2c\sigma^2\right)^{-\frac{m}{2}} \qquad \text{for } c < \frac{1}{2\sigma^2}.$$

Note that the OLS, ML, and REML estimators for β are identical. Furthermore, the OLS estimator for σ^2 is $\widehat{\sigma}^2_{REML} = \text{RSS}/m$, which is the same as the REML estimator. The ML estimator for σ^2 is $\widehat{\sigma}^2_{ML} = \text{RSS}/n$.

2.1 *The naive back-transform estimator*

Since the UMVUE of $\log(Z_0)$ is $x_0^T\hat{\beta}$, it may seem reasonable to estimate $\mu(x_0)$ by $\widehat{\mu}_{BT}(x_0) = \exp\left(x_0^T\hat{\beta}\right)$, the back-transform (BT) estimator. This is also a commonly used estimator in practice. However, by comparing it to (2), it is easy to show that the BT estimator is not even consistent. Even when the sample size is large, this estimator has an asymptotic multiplicative bias of $\exp(-\sigma^2/2)$, which is always less than one. As a result, the BT estimator underestimates $\mu(x_0)$, and the bias is large when σ^2 is large. In our study, it appears that the BT estimator performs much worse than the other estimators. Thus, we do not include it in our comparison below in Section 4. Actually, the BT estimator is more suitable for estimating the median of Z_0, which is $\exp\left(x_0^T\beta\right)$ in this case.

2.2 *The ML/REML estimators*

The ML and REML estimators of $\mu(x_0)$ are given by

$$\widehat{\mu}_{ML}(x_0) = \exp\left(x_0^T\hat{\beta} + \widehat{\sigma}^2_{ML}/2\right)$$

and

$$\widehat{\mu}_{REML}(x_0) = \exp\left(x_0^T\hat{\beta} + \widehat{\sigma}^2_{REML}/2\right)$$

respectively. The MSE and bias of $\widehat{\mu}_{ML}(x_0)$ can be derived using the results in Propositions 1 and 2 as follows,

$$\begin{aligned} & \mathrm{MSE}[\widehat{\mu}_{ML}(x_0)] \qquad (4) \\ = \; & \mu^2(x_0)\left[e^{(2v_0-1)\sigma^2}(1-2\sigma^2/n)^{-\frac{m}{2}} - 2e^{(v_0-1)\sigma^2/2}(1-\sigma^2/n)^{-m/2} + 1\right], \end{aligned}$$

and

$$\mathrm{Bias}[\widehat{\mu}_{ML}(x_0)] = \mu(x_0)\left[e^{\frac{1}{2}(v_0-1)\sigma^2}(1-\sigma^2/n)^{-\frac{m}{2}} - 1\right]. \qquad (5)$$

The MSE and bias of $\widehat{\mu}_{REML}(x_0)$ can be obtained by replacing n in (4) and (5) with m.

2.3 *The UMVU estimator*

Bradu and Mundlak (1970) derived the UMVU estimator of $\mu(x_0)$ using the fact that $\hat{\beta}$ and $\hat{\sigma}^2_{REML}$ are complete sufficient statistics for β and σ^2, and any unbiased function of the complete sufficient statistics is the UMVU estimator of the mean of that function (Lehmann and Casella, 1998). To obtain the UMVU estimator of $\mu(x_0)$, one only needs to find $f(x)$ such that

$$\mathrm{E}\left[e^{x_0^T\hat{\beta}}f(\hat{\sigma}^2_{REML})\right] = \exp\left(x_0^T\beta + \frac{1}{2}\sigma^2\right),$$

which leads to

$$\begin{aligned} f(\hat{\sigma}^2_{REML}) &= \sum_{i=0}^{\infty}\frac{\Gamma(m/2)}{i!\Gamma(m/2+i)}\left[\frac{m(1-v_0)}{4}\hat{\sigma}^2_{REML}\right]^i \\ &= {}_0F_1\left(\frac{m}{2};\frac{m(1-v_0)}{4}\hat{\sigma}^2_{REML}\right), \end{aligned}$$

where ${}_0F_1(\alpha;z)$ is the Hypergeometric function (Seaborn, 1991). The UMVU estimator and its variance are given by

$$\widehat{\mu}_{UMVU}(x_0) = e^{x_0^T\hat{\beta}}{}_0F_1\left(\frac{m}{2};\frac{m(1-v_0)}{4}\hat{\sigma}^2_{REML}\right), \tag{6}$$

and

$$\mathrm{Var}[\widehat{\mu}_{UMVU}(x_0)] = \mu^2(x_0)\left[e^{v_0\sigma^2}{}_0F_1\left(\frac{m}{2};\frac{(1-v_0)}{4}\sigma^4\right) - 1\right]. \tag{7}$$

More details can be found in Finney (1941) and Bradu and Mundlak (1970). Note that the UMVU estimator is an unbiased estimator, i.e., the bias is zero, which means that its variance is the same as its MSE. The UMVU estimator has the smallest MSE among all unbiased estimators.

2.4 *The bias-corrected REML estimator*

It is well known that the ML/REML estimators exhibit some bias, and El-shaarawi and Viveros (1997) proposed to correct this bias using the leading terms of the Taylor expansion of Bias$[\widehat{\mu}_{REML}(x_0)]$ with respect to σ^2/m, which leads to the following bias-corrected REML estimator which we refer to as the EV estimator,

$$\widehat{\mu}_{EV}(x_0) = \exp\left[x_0^T\hat{\beta} + \frac{(1-v_0)}{2}\hat{\sigma}^2_{REML} - \frac{1}{4m}\hat{\sigma}^4_R - \frac{1}{6m}\hat{\sigma}^6_R\right]. \qquad (8)$$

Its MSE and bias are given by

$$\begin{aligned}&\mathrm{MSE}[\widehat{\mu}_{EV}(x_0)]\\ =\;& \mu^2(x_0)\left\{e^{(2v_0-1)\sigma^2}\mathrm{E}\left[f^2_{EV}(\hat{\sigma}^2_{REML})\right] - 2e^{(v_0-1)\sigma^2/2}\mathrm{E}\left[f_{EV}(\hat{\sigma}^2_{REML})\right] + 1\right\},\end{aligned}$$

and

$$\mathrm{Bias}[\widehat{\mu}_{EV}(x_0)] = \mu(x_0)\left\{e^{(v_0-1)/2\sigma^2}\mathrm{E}\left[f_{EV}(\hat{\sigma}^2_{REML})\right] - 1\right\},$$

where

$$f_{EV}(x) = \exp\left[\frac{(1-v_0)}{2}x - \frac{1}{4m}x^2 - \frac{1}{6m}x^3\right],$$

and the two expectations have to be evaluated using numerical integration.

3. Two new estimators and their confidence intervals

In this section, we propose two new estimators. The first estimator minimizes the MSE (3) approximately, and is defined as,

$$\widehat{\mu}_{MM}(x_0) = \exp\left[x_0^T\widehat{\beta} + \frac{m\mathrm{RSS}}{2(n-p+1+3nv_0)m + 3\mathrm{RSS}}\right],$$

where $m = n-(p+1)$. The second estimator, on the other hand, minimizes the bias considerably, and is defined as,

$$\widehat{\mu}_{MB}(x_0) = \exp\left[x_0^T\widehat{\beta} + \frac{m\mathrm{RSS}}{2(n-p-1+nv_0)m + \mathrm{RSS}}\right].$$

The proposed estimators can be viewed as degree-of-freedom-adjusted ML estimators. In practice, it is very easy to obtain these estimators, because $\widehat{\beta}$ and RSS can be readily calculated. Below we describe how the estimators are derived.

3.1 *Derivation of the estimators*

In light of the special relationship (2), we propose to look at the following class of estimators,

$$\left\{\widehat{\mu}_c(x_0):\ \widehat{\mu}_c(x_0)=\exp\left(x_0^T\widehat{\beta}+c\text{RSS}/2\right),\ c=\frac{1}{n-a},\ a<n\right\}. \qquad (9)$$

Intuitively, this class of estimators are of simple form, and can be described as plug-in estimators relative to the basic formula (2) with $\widehat{\beta}$ and $c\text{RSS} = \text{RSS}/(n-a)$ serving as the estimators of β and σ^2 respectively. The class of estimators are asymptotically equivalent and efficient, because the ML estimator belongs to the class with $a = 0$.

Our goal is to find estimators from this class that can asymptotically minimize the MSE or the bias, and have better or comparable performances as the existing estimators mentioned in Section 2.

THEOREM 1. *Under the condition that $c < \frac{1}{2\sigma^2}$, the MSE of $\widehat{\mu}_c(x_0)$ is*

$$\begin{aligned} &MSE[\widehat{\mu}_c(x_0)] = E[\widehat{\mu}_c(x_0)-\mu(x_0)]^2 \\ = \ &\mu^2(x_0)\left[e^{(2v_0-1)\sigma^2}(1-2c\sigma^2)^{-\frac{m}{2}} - 2e^{\frac{1}{2}(v_0-1)\sigma^2}(1-c\sigma^2)^{-\frac{m}{2}}+1\right]. \end{aligned}$$

In addition, the bias of $\widehat{\mu}_c(x_0)$ is

$$Bias[\widehat{\mu}_c(x_0)] = E[\widehat{\mu}_c(x_0)-\mu(x_0)] = \mu(x_0)\left[e^{\frac{1}{2}(v_0-1)\sigma^2}(1-c\sigma^2)^{-\frac{m}{2}}-1\right].$$

Theorem 1 can be proved by making use of the MGF of RSS as stated in Proposition 2. The theorem suggests that a direct minimization of MSE $[\widehat{\mu}_c(x_0)]$ seems implausible as a path to a convenient and satisfactory procedure. As an alternative, we look at the second order asymptotics to find a constant c that can asymptotically minimize the MSE. A similar approach is employed in Shen et al. (2005) to derive an efficient estimator for one-population log-normal means.

Note the following standard expansion,

$$c = \frac{1}{n-a} = \frac{1}{n} + \frac{a}{n^2} + \frac{a^2}{n^3} + o\left(\frac{1}{n^3}\right),$$

which leads us to consider estimators of the form $\widehat{\mu}_c(x_0)$ with $c = 1/n + a/n^2 + b/n^3 + o(1/n^3)$.

THEOREM 2. *Suppose* $c = 1/n + a/n^2 + b/n^3 + o(1/n^3)$. *Then,*

$$\begin{aligned} & MSE\,[\widehat{\mu}_c(x_0)] \\ = \;& \mu^2(x_0)\frac{\sigma^2}{n}\left\{1 + \frac{\sigma^2}{2} + \frac{\sigma^2}{4n}\left[a^2 + \left(2 - 2p + 6nv_0 + 3\sigma^2\right)a + f\left(p, n, \sigma^2, v_0\right)\right]\right\} \\ & + o\left(\frac{1}{n^2}\right), \end{aligned}$$

where $f\left(p, n, \sigma^2, v_0\right) = -1 + p^2 - 6nv_0(p+1) + 7n^2v_0^2 + (1 - 3p + 7nv_0)\sigma^2 + 7\sigma^4/4$;

$$Bias\,[\widehat{\mu}_c(x_0)] = \mu(x_0)\frac{\sigma^2}{2n}\left(nv_0 + a - p - 1 + \frac{\sigma^2}{2}\right) + o\left(\frac{1}{n}\right).$$

The proof of Theorem 2 is provided in the Appendix. Note that the above expansions do not depend on the second constant, b.

Suppose one wants to find a constant c that can minimize the MSE up to the order of $1/n^2$. Theorem 2 suggests that it suffices to find a to minimize

$$a^2 + \left(2 - 2p + 6nv_0 + 3\sigma^2\right)a.$$

According to the quadratic form, the minimizer depends on σ^2 and is

$$-(1 - p + 3nv_0 + 3\sigma^2/2).$$

This means that the constant c which minimizes the approximate MSE should be of the order of $1/(n + 1 - p + 3nv_0 + 3\sigma^2/2)$. This is thus the value an oracle would choose. However, in real applications, the true variance σ^2 is usually unknown. We propose to use an "adaptive" estimator by replacing σ^2 with its consistent estimator, $\widehat{\sigma}^2_{REML} = \mathrm{RSS}/m$. As a result, our proposed estimator is

$$\widehat{\mu}_{MM}(x_0) = \exp\left[x_0^T\widehat{\beta} + \frac{m\mathrm{RSS}}{2(n-p+1+3nv_0)m + 3\mathrm{RSS}}\right].$$

On the other hand, suppose one wants to find a constant c to reduce the bias to the order of $1/n$. According to Theorem 2, it suffices to find a to satisfy

$$nv_0 + a - p - 1 + \frac{\sigma^2}{2} = 0,$$

which leads to

$$a = p + 1 - nv_0 - \sigma^2/2.$$

This means that the constant c which minimizes the approximate bias should be of the order of $1/(n - p - 1 + nv_0 + \sigma^2/2)$. Similarly, we propose to use an "adaptive" estimator by replacing σ^2 with its consistent estimator, $\widehat{\sigma}^2_{REML}$. As a result, our proposed estimator is

$$\widehat{\mu}_{MB}(x_0) = \exp\left[x_0^T\widehat{\beta} + \frac{m\mathrm{RSS}}{2(n-p-1+nv_0)m + \mathrm{RSS}}\right].$$

The MSE and bias of the two proposed estimators are summarized in the following corollary.

Corollary 1. *Let*

$$f_{MM}\left(RSS\right)=\exp\left[\frac{mRSS}{2(n-p+1+3nv_0)m+3RSS}\right]$$

and

$$f_{MB}\left(RSS\right)=\exp\left[\frac{mRSS}{2(n-p+1+nv_0)m+RSS}\right].$$

Then,

$$\begin{aligned}&MSE\left[\widehat{\mu}_{MM}(x_0)\right]\\ =\;&\mu^2(x_0)\left[e^{(2v_0-1)\sigma^2}E\left(f_{MM}^2(RSS)\right)-2e^{\frac{1}{2}(v_0-1)\sigma^2}E\left(f_{MM}(RSS)\right)+1\right],\end{aligned}$$

$$Bias\left[\widehat{\mu}_{MM}(x_0)\right]=\mu(x_0)\left[e^{\frac{1}{2}(v_0-1)\sigma^2}E\left(f_{MM}(RSS)\right)-1\right];$$

$$\begin{aligned}&MSE\left[\widehat{\mu}_{MB}(x_0)\right]\\ =\;&\mu^2(x_0)\left[e^{(2v_0-1)\sigma^2}E\left(f_{MB}^2(RSS)\right)-2e^{\frac{1}{2}(v_0-1)\sigma^2}E\left(f_{MB}(RSS)\right)+1\right],\end{aligned}$$

$$Bias\left[\widehat{\mu}_{MB}(x_0)\right]=\mu(x_0)\left[e^{\frac{1}{2}(v_0-1)\sigma^2}E\left(f_{MB}(RSS)\right)-1\right].$$

In Section 4 we will compare the MSE and bias of our estimators $\widehat{\mu}_{MM}(x_0)$ and $\widehat{\mu}_{MB}(x_0)$ with the existing estimators described in Section 2.

Four aforementioned estimators (ML, REML, MM, and MB) belong to the class of estimators defined in (9), with $a_{ML}=0$, $a_{REML}=p+1$, $a_{MM}=p-1-3nv_0-3\text{RSS}/(2m)$, and $a_{MB}=p+1-nv_0-\text{RSS}/(2m)$. According to Theorem 2, both the MSE and the Bias^2 are asymptotically quadratic functions of a when ignoring the $o\left(1/n^2\right)$ terms. Figure 1 plots the MSE and Bias^2 as functions of a for $p=1$, $n=50$, x taking values that are uniformly distributed between 0 and 1, $\beta=(1,1)^T$, $\sigma^2=1$, and $x_0=(1,0.5)^T$. The a_{MM} and a_{MB} are marked by the solid and dotted vertical lines respectively, which correspond to the two estimators that minimize the MSE and the Bias^2 respectively. In practice, one may want a different trade-off between the MSE and the Bias^2, and in principle one should choose a value of a between a_{MM} and a_{MB} to construct an estimator for that particular purpose. On the other hand, it is clear from Figure 1 that any value of a outside the interval of $[a_{MM}, a_{MB}]$ is worse than either the MM estimator or the MB estimator in terms of both MSE and Bias^2, thus those values should never be used. Since $v_0 \geq 0$, it is easy to show that $a_{REML} > a_{MM}$ and $a_{REML} > a_{MB}$. Consequently, we conclude that the REML estimator should not be used in practice. The numerical studies in Section 4 also confirm this conclusion. The a_{ML}, on the other hand, may fall between a_{MM} and a_{MB} for some combinations of x_0, σ^2, n, and p.

[Figure 1 about here.]

To define the MB estimator, we let $c = 1/(n - a)$ and choose a such that the bias term is of the order of $o\,(1/n)$. We can actually do better than that by letting $c = 1/n + a/n^2 + b/n^3$, and choose a and b such that $\text{Bias}[\widehat{\mu}_c(x_0)] = o(1/n^2)$. See (A.2) in the Appendix for details. The bias of this estimator is then of higher order than $1/n^2$, and we will refer to it as the super minimum bias (SMB) estimator. Numerical studies indicate that the SMB estimator indeed has smaller bias than the MB estimator, but the difference is negligible for all practical purposes. Thus we do not present its MSE and bias here.

3.2 *Parametric Bootstrap Confidence Intervals*

For statistical inference purpose, it makes sense to investigate confidence intervals for the lognormal mean, $\mu(x_0)$. The relation (2) suggests that confidence intervals for $\mu(x_0)$ can be derived by exponentiating confidence intervals for $\tau(x_0) = \log[\mu(x_0)] = x_0^T\beta + \frac{1}{2}\sigma^2$. In this section, we first propose a general procedure to derive parametric bootstrap confidence intervals for $\tau(x_0)$ around an arbitrary estimator of the following form,

$$\widehat{\tau}(x_0) = \log[\widehat{\mu}(x_0)] = x_0^T\widehat{\beta} + g(\mathrm{RSS}),$$

which then leads to a confidence interval for $\mu(x_0)$ around $\widehat{\mu}(x_0)$. Then, confidence intervals for $\widehat{\mu}_{MM}(x_0)$ and $\widehat{\mu}_{MB}(x_0)$ can be derived as special cases. Our simulation study in Section 4 suggests that the proposed confidence intervals have nice coverage properties.

We know that $x_0^T\widehat{\beta} \sim N\left(x_0^T\beta, v_0\sigma^2\right)$ and $\mathrm{RSS} \sim \sigma^2\chi_m^2$. Then, using the Delta method, we can obtain the following approximate expression for the variance of $\widehat{\tau}(x_0)$,

$$\mathrm{Var}[\widehat{\tau}(x_0)] \approx v_0\sigma^2 + 2m\sigma^4 g'^2(m\sigma^2).$$

Note that σ^2 can be estimated using RSS/m. Define the following statistic,

$$K[\widehat{\tau}(x_0)] = \frac{\widehat{\tau}(x_0) - \tau(x_0)}{\sqrt{\widehat{\mathrm{Var}}[\widehat{\tau}(x_0)]}} = \frac{x_0^T\widehat{\beta} + g(\mathrm{RSS}) - \tau(x_0)}{\sqrt{v_0\mathrm{RSS}/m + 2mg'^2(m\mathrm{RSS}/m)(\mathrm{RSS}/m)^2}}. \tag{10}$$

For a significance level α, let t_1 and t_2 be the $\alpha/2$ and $1-\alpha/2$ percentiles of $K[\widehat{\tau}(x_0)]$, respectively. Then, one can obtain a $1-\alpha$ confidence interval for $\tau(x_0)$ as

$$\left[\widehat{\tau}(x_0) - t_2\sqrt{\widehat{\mathrm{Var}}[\widehat{\tau}(x_0)]}, \widehat{\tau}(x_0) - t_1\sqrt{\widehat{\mathrm{Var}}[\widehat{\tau}(x_0)]}\right].$$

To estimate the two percentiles, we observe from (10) that $K\left[\widehat{\tau}(x_0)\right]$ has the same distribution as

$$T(\sigma) = \frac{N + \frac{\sigma}{2\sqrt{v_0}}\left[\frac{2}{\sigma^2} g\left(m\sigma^2 C_m\right) - 1\right]}{\sqrt{C_m + \frac{2m}{v_0\sigma^2} g'^2\left(m\sigma^2 C_m\right)\left(\sigma^2 C_m\right)^2}}, \tag{11}$$

where $N \sim N(0,1)$, $C_m \sim \chi_m^2/m$ and they are independent. Thus we propose the following parametric bootstrap procedure to estimate t_1 and t_2:

1. Generate $N_i \sim N(0,1)$ and $C_i \sim \chi_m^2/m$ independently for $i = 1, \ldots, B$;
2. Calculate T_i according to (11) with N, C and σ replaced with N_i, C_i and $\sqrt{\text{RSS}/m}$;
3. Estimate t_1 by $\hat{t}_1$, the $\alpha/2$ percentile of $\{T_i : i = 1, \ldots, B\}$, and t_2 by $\hat{t}_2$, the $1-\alpha/2$ percentile of the T_i's.

As a result, we obtain a $1-\alpha$ parametric bootstrap confidence interval for $\tau(x_0)$ as

$$\left[\widehat{\tau}(x_0) - \hat{t}_2\sqrt{\widehat{\text{Var}}\left[\widehat{\tau}(x_0)\right]}, \widehat{\tau}(x_0) - \hat{t}_1\sqrt{\widehat{\text{Var}}\left[\widehat{\tau}(x_0)\right]}\right]. \tag{12}$$

Then, the corresponding $1-\alpha$ bootstrap confidence interval for $\mu(x_0)$ is

$$\exp\left[\widehat{\tau}(x_0) - \hat{t}_2\sqrt{\widehat{\text{Var}}\left[\widehat{\tau}(x_0)\right]}, \widehat{\tau}(x_0) - \hat{t}_1\sqrt{\widehat{\text{Var}}\left[\widehat{\tau}(x_0)\right]}\right]. \tag{13}$$

In particular, our two estimators correspond to the following g functions,

$$g_{MM}\left(\text{RSS}\right) = \log\left[f_{MM}\left(\text{RSS}\right)\right] = \frac{m\text{RSS}}{2(n-p+1+3nv_0)m + 3\text{RSS}},$$

and

$$g_{MB}\left(\text{RSS}\right) = \log\left[f_{MB}\left(\text{RSS}\right)\right] = \frac{m\text{RSS}}{2(n-p-1+nv_0)m + \text{RSS}}.$$

We can use the above general parametric bootstrap procedure to generate the corresponding confidence intervals for our proposed estimators.

Alternatively, one can assume that $\widehat{\tau}(x_0)$ is normally distributed and derive an approximate $1-\alpha$ variance confidence interval for $\mu(x_0)$ as

$$\exp\left[\widehat{\tau}(x_0) - z_{1-\alpha/2}\sqrt{\widehat{\text{Var}}\left[\widehat{\tau}(x_0)\right]}, \widehat{\tau}(x_0) + z_{1-\alpha/2}\sqrt{\widehat{\text{Var}}\left[\widehat{\tau}(x_0)\right]}\right], \qquad (14)$$

where $z_{1-\alpha/2}$ is the $(1-\alpha/2)$-percentile of the standard normal distribution. In Section 4, the coverage properties of the two confidence intervals are investigated via a simulation study.

4. Simulation study

4.1 *Numerical comparison of the estimators*

In this section we numerically compute the MSE and bias of the MM and MB estimators for some combinations of n and σ^2 which are typical in many applications, and compare them with the other estimators mentioned in Section 2 to illustrate how different estimators perform under different scenarios. We assume that the data can be modeled as in (1), with only one covariate x, taking values that are uniformly distributed between 0 and 1. The regression coefficient vector $\beta = (\beta_0, \beta_1)^T$ is taken to be $(1,1)^T$. Extension to scenarios with multiple covariates is simple. We present our results for $\sigma^2 \in \{0.25, 0.5, 1\}$ and sample size $n \in \{10, 50\}$, where σ^2 is the variance of ϵ_i. σ^2 larger than one is unlikely to occur in real applications, and sample sizes larger than 50 yield qualitatively similar results as the case of $n = 50$. Those results are not reported here. We consider the estimation of $\mu(x_0)$ for $x_0 \in \{0, 0.1, \ldots, 1.2\}$.

[Figure 2 about here.]

For different combinations of n and σ^2, Figure 2 plots the relative MSE of the ML, REML, UMVU, EV, MB, and MM estimators, which is the ratio between the MSEs of a particular estimator and the MM estimator. The relative MSE of the MM estimator is always one by construction, and the values for the other estimators represent their MSEs as percentages of the MSE of the MM estimator. The following patterns are observed from the plots:

1. The REML estimator is worse than all the other estimators across the whole range of x_0 and all the considered combinations, and the difference is quite large for a small sample size ($n = 10$) and a large σ^2 ($\sigma^2 = 1$). This is consistent with what we found in Section 3, and reaffirmed our claim that the REML estimator should not be used.

2. The MSE of the ML estimator is comparable to the UMVU, EV, and MB estimators when x_0 is close to the center of the data used to fit the regression model, but it increases much faster than the other estimators when x_0 is away from the center. Thus one needs to be extra cautious when using the ML estimator for extrapolation.

3. The MM estimator has the smallest MSE among all the estimators for all the cases we considered, and the difference in some cases is rather substantial. For example, for a small sample size ($n = 10$), the MSE of the UMVU estimator is about 8% larger for $\sigma^2 = 0.25$, and about 40% larger for $\sigma^2 = 1$. Even when the sample size is relatively large ($n = 50$), the MSE of the UMVU estimator is still more than 10% larger when $\sigma^2 = 1$. These results show that the MM estimator can be much more efficient in terms of MSE, especially when the sample size is not large.

4. The MSE of the MB estimator is very close to that of the UMVU estimator, especially when n is large. Since the bias of the MB estimator is also very small (see Figure 3 and its comment 2), in practice one can use the MB estimator as a surrogate for the UMVU estimator, which is much harder to compute.

[Figure 3 about here.]

Below we look at the bias for the three estimators constructed to reduce bias, the EV, MB, and SMB estimators. In Figure 3, the relative absolute bias of these three estimators are plotted. The relative absolute bias is defined as the absolute bias of an estimator of $\mu(x_0)$ divided by $\mu(x_0)$. Thus it has the nice interpretation as being the bias in terms of the percentage of the estimand. The following comments can be made about the plots:

1. All three estimators have rather small relative bias. In the worst case ($n = 10$, $\sigma^2 = 1$, and $x_0 = 0$), the largest relative bias for the EV estimator is barely larger than 1%. In all the cases we considered, the relative bias of the MB and SMB estimators are below 0.2%.

2. Except for the case when $\sigma^2 = 0.25$, the MB estimator has smaller bias than the EV estimator. The SMB estimator has the smallest bias over most of the range of x_0. However, the difference between the MB estimator and the SMB estimator is too small to justify the use of the more complicated SMB estimator.

4.2 *Coverage Properties of the Bootstrap and Variance Confidence Intervals*

In this section, we use the same setup as in Section 4.1 to investigate the coverage properties of the bootstrap and variance confidence intervals for the

MM and MB estimators. The two confidence intervals are defined in (13) and (14), respectively. For each combination of sample size and σ^2, 5000 confidence intervals are derived for each estimator at each x_0. The empirical coverage probability is calculated as the proportion of these confidence intervals that cover the true value. To derive a bootstrap confidence interval, we set B to be 5000 as well when using the parametric bootstrap procedure of Section 3.2.

[Figure 4 about here.]

Figure 4 compares the empirical coverage probabilities for the confidence intervals corresponding to the two estimators. The nominal coverage level (95%) is also superimposed as the benchmark. A number of patterns emerge from the plots:

1. In general, the bootstrap confidence intervals have better coverage properties than the variance confidence intervals. The bootstrap confidence intervals perform reasonably well with coverage probabilities ranging between 92% and 96%. On the other hand, the variance intervals have coverage probabilities ranging between 83% and 95%, and undercover considerably for a small sample size, especially when σ^2 is large.

2. The bootstrap confidence intervals for the two estimators have comparable coverage properties. The average coverage probabilities across the x range are 94.4% and 94.6%, respectively. We also compare the lengths of the confidence intervals, and find that the MM estimator leads to shorter intervals in all cases. The average ratio ranges between 96.0% and 99.7%.

3. The variance confidence intervals for the MB estimator have higher coverage probabilities than those for the MM estimator (90.4% versus 92.7%).

Thus, we recommend to use the bootstrap confidence intervals, especially in scenarios where a small sample size and a large σ^2 might occur. The MM bootstrap confidence interval is preferred for a tighter interval.

5. Application to a Sediment discharge data

The study of sediment transport is of interest to many people because of its environmental impacts (Bollman, 1992 and Cohn, 1995). Estimates of suspended-sediment loads are often derived from instantaneous streamflows using lognormal regression models, where both sediment loads and streamflows are transformed into the logarithmic scale. The final estimates of sediment loads, however, are often required to be in the original scale. On the U.S. Geological Survey (USGS) website, a small data set on the streamflow (Q, in the unit of m^3/s) and sediment discharge (L, in the unit of metric tons) is analyzed to compare several mean estimation methods for bias correction. Here we use the same dataset to illustrate our estimation methods.

Table 1 gives the streamflow and sediment discharge data collected at a gaging station. A log-log normal regression is believed to be the correct model (USGS website), and the fitted model is

$$\log(L) = -3.90 + 2.76 \log(Q),$$

with $\hat{\sigma}^2_{REML} = 0.346$.

[Table 1 about here.]

[Table 2 about here.]

[Figure 5 about here.]

Table 2 gives the daily values of streamflow for a 14-day study period. The problem of interest here is to estimate the daily sediment discharge during this 14-day period. We compute the ML, UMVU, EV, MM, and MB estimates and their theoretical bias and MSE for these 14 days, and plot the relative absolute bias and the relative MSE as functions of $\log(Q)$ in Figure 5. The MB estimator has the smallest bias (less than 1%) for almost all streamflow values during the study period among all the estimators other than the UMVU estimator, which has no bias by construction. The MSEs of the MB estimator are very close to those of the UMVU estimator. Since the MB estimator can be easily computed using the LS estimators of β and σ^2, it can be used as a surrogate of the UMVU estimator in practice when no software is handy to evaluate the Hypergeometric function. The MM estimator has the smallest MSE among all estimators, including the UMVU estimator.

The MM estimates are slightly smaller than the MB estimates, and the average ratio is 94.5%. We set B to be 5000 when generating the 95% bootstrap confidence intervals for these two estimates. Except for the second observation, the MM estimator leads to narrower confidence intervals, which are on average about 95.3% of the lengths for the MB confidence intervals. Thus, the intervals based on the MM estimator might be preferred if one wants narrower confidence intervals instead of unbiased point estimates. We only report the MM estimates along with their bootstrap confidence intervals in Table 2.

6. Conclusion

In this paper, we have proposed two new estimators for the conditional mean of the response variable in the original scale for lognormal linear models, the MM estimator and the MB estimator. Both estimators can be viewed as "degree-of-freedom-adjusted" ML estimators, which are simple functions of the ML estimator of the regression coefficient β and the RSS from the lognormal linear models, and hence are very easy to compute. Comparisons with the existing estimators show that the MB estimator has almost identical performance as the UMVU estimator, while is much easier to compute. It thus can be used as a simple alternative to the UMVU estimator. The MM estimator has smaller MSEs than the other estimators for the scenarios considered, and the gain in efficiency can be quite substantial when the sample size is small and σ^2 is moderately large. Its use is recommended if one wants to minimize the square error risk. A parametric bootstrap procedure is also proposed to produce confidence intervals for our estimators, which have a small coverage bias. Among the existing estimators, we conclude that the usual REML estimator should never be used, and our simulation studies suggest that the EV estimator, i.e. the bias-corrected REML estimator, often achieves a reasonable balance between bias and variance, even though its bias is slightly larger than our MB estimator, and its MSE is larger than our MM estimator.

The current study assumes that the errors in the lognormal linear models are independent and have an equal variance. In practice both assumptions may be violated. It is relatively straight forward to adapt our approach to take into account possible inhomogeneity in the error variance. However, it is more difficult to adjust for correlated errors, which are often present in time series and spatial data sets, and deserve further investigation. We intend to address this issue in a separate manuscript.

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Appendix A
Proof of Theorem 2

First notice the following Taylor series expansion,

$$\log(1-t) = -\sum_{i=1}^{\infty} \frac{t^i}{i}.$$

Define $V_1 = \exp\left[(2v_0 - 1)\sigma^2\right](1-2c\sigma^2)^{-m/2}$ and $V_2 = \exp\left[\frac{1}{2}(v_0 - 1)\sigma^2\right](1-c\sigma^2)^{-m/2}$. Below we expand V_1 and V_2 using the above Taylor expansion.

$$
\begin{aligned}
V_1 &= e^{\left[(2v_0-1)\sigma^2-\frac{m}{2}\log(1-2c\sigma^2)\right]} \\
&= e^{\left[(2v_0-1)\sigma^2+\frac{m}{2}\left(2c\sigma^2+2c^2\sigma^4+\frac{8}{3}c^3\sigma^6+o\left(\frac{1}{n^3}\right)\right)\right]} \\
&= e^{\left[\frac{2n^2v_0+n(a-p-1)+b-a(p+1)}{n^2}\sigma^2+\left(\frac{1}{n}+\frac{2a-1}{n^2}\right)\sigma^4+\frac{4}{3n^2}\sigma^6+o\left(\frac{1}{n^2}\right)\right]} \\
&= 1+\left[2v_0+\frac{a-p-1}{n}+\frac{b-a(p+1)}{n^2}\right]\sigma^2+\left(\frac{1}{n}+\frac{2a-p-1}{n^2}\right)\sigma^4+\frac{4}{3n^2}\sigma^6 \\
&\quad +\frac{(2nv_0+a-p-1)^2}{2n^2}\sigma^4+\frac{1}{2n^2}\sigma^8+\frac{2nv_0+a-p-1}{n^2}\sigma^6+o\left(\frac{1}{n^2}\right) \\
&= 1+\left(2nv_0+a-p-1+\sigma^2\right)\frac{\sigma^2}{n}+[b-a(p+1)]\frac{\sigma^2}{n^2} \\
&\quad +\left[2a-p-1+2n^2v_0^2+2nv_0(a-p-1)+\frac{1}{2}(a-p-1)^2\right]\frac{\sigma^4}{n^2} \\
&\quad +\left(\frac{4}{3}+2nv_0+a-p-1\right)\frac{\sigma^6}{n^2}+\frac{1}{2n^2}\sigma^8+o\left(\frac{1}{n^2}\right).
\end{aligned}
$$

$$
\begin{aligned}
V_2 &= e^{\left[\frac{1}{2}(v_0-1)\sigma^2-\frac{m}{2}\log(1-c\sigma^2)\right]} \qquad \text{(A.1)} \\
&= e^{\left[\frac{1}{2}(v_0-1)\sigma^2+\frac{m}{2}\left(c\sigma^2+\frac{c^2\sigma^4}{2}+\frac{c^3\sigma^6}{3}+o\left(\frac{1}{n^3}\right)\right)\right]} \\
&= e^{\left[\frac{n^2v_0+n(a-p-1)+b-a(p+1)}{2n^2}\sigma^2+\frac{n+2a-p-1}{4n^2}\sigma^4+\frac{1}{6n^2}\sigma^6+o\left(\frac{1}{n^2}\right)\right]} \\
&= 1+\frac{n^2v_0+n(a-p-1)+b-a(p+1)}{2n^2}\sigma^2+\frac{n+2a-p-1}{4n^2}\sigma^4+\frac{1}{6n^2}\sigma^6 \\
&\quad +\frac{(nv_0+a-p-1)^2}{8n^2}\sigma^4+\frac{1}{32n^2}\sigma^8+\frac{nv_0+a-p-1}{8n^2}\sigma^6+o\left(\frac{1}{n^2}\right) \\
&= 1+\left(nv_0+a-p-1+\frac{\sigma^2}{2}\right)\frac{\sigma^2}{2n}+[b-a(p+1)]\frac{\sigma^2}{2n^2} \\
&\quad +\left[a-\frac{p+1}{2}+\frac{n^2v_0^2}{4}+\frac{nv_0(a-p-1)}{2}+\frac{(a-p-1)^2}{4}\right]\frac{\sigma^4}{2n^2} \\
&\quad +\left(\frac{1}{3}+\frac{nv_0}{4}+\frac{a-p-1}{4}\right)\frac{\sigma^6}{2n^2}+\frac{\sigma^8}{32n^2}+o\left(\frac{1}{n^2}\right).
\end{aligned}
$$

According to Theorem 1, we know that

$$
\mathrm{MSE}\left[\widehat{\mu}_c(x_0)\right]=\mu^2(x_0)\left(V_1-2V_2+1\right).
$$

Thus, incorporating the above expressions for V_1 and V_2, we obtain

$$\begin{aligned}
& \text{MSE}\left[\widehat{\mu}_c(x_0)\right] \\
= \ & \mu^2(x_0)\frac{\sigma^2}{n}\left\{1+\frac{\sigma^2}{2}+\frac{\sigma^2}{4n}\left[a^2+\left(2-2p+6nv_0+3\sigma^2\right)a+f\left(p,n,\sigma^2,v_0\right)\right]\right\} \\
& +o\left(\frac{1}{n^2}\right),
\end{aligned}$$

where $f\left(p,n,\sigma^2,v_0\right)=-1+p^2-6nv_0(p+1)+7n^2v_0^2+(1-3p+7nv_0)\sigma^2+7\sigma^4/4$.

From Theorem 1, we also know that

$$\text{Bias}\left[\widehat{\mu}_c(x_0)\right]=\mu(x_0)\left(V_2-1\right).$$

Thus, incorporating the above expression for V_2, we obtain

$$\text{Bias}\left[\widehat{\mu}_c(x_0)\right]=\mu(x_0)\frac{\sigma^2}{2n}\left(nv_0+a-p-1+\frac{\sigma^2}{2}\right)+o\left(\frac{1}{n}\right).$$

In addition, if we keep one more term for Bias $\left[\widehat{\mu}_c(x_0)\right]$, then

$$\text{Bias}\left[\widehat{\mu}_c(x_0)\right]=\mu(x_0)\left(\frac{\sigma^2}{2n}c_1+\frac{\sigma^2}{2n^2}c_2\right)+o\left(\frac{1}{n^2}\right), \tag{A.2}$$

where

$$c_1=nv_0+a-p-1+\frac{1}{2}\sigma^2$$

and

$$\begin{aligned}
c_2 \ = \ & b-a(p+1)+\left[a-\frac{p+1}{2}+\frac{n^2v_0^2}{4}+\frac{nv_0(a-p-1)}{2}+\frac{(a-p-1)^2}{4}\right]\sigma^2 \\
+ \ & \left(\frac{1}{3}+\frac{nv_0}{4}+\frac{a-p-1}{4}\right)\sigma^4+\frac{1}{16}\sigma^6.
\end{aligned}$$

Thus, one can define an estimator by choosing a and b such that c_1 and c_2 equal to 0. See a relevant discussion at the end of Section 3.1. □

Figure 1. The MSE and Bias^2 as functions of a.

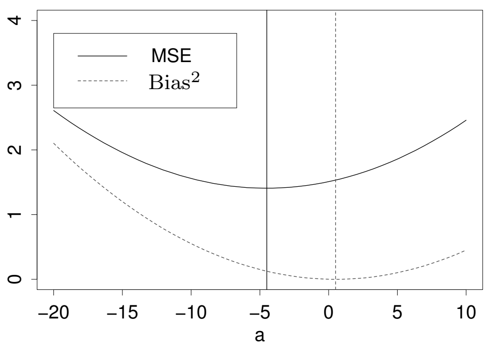

Figure 2. Relative MSE for different estimators, which is obtained by dividing the MSE of each estimator by the MSE of the MM estimator. The MM estimator has the smallest MSE.

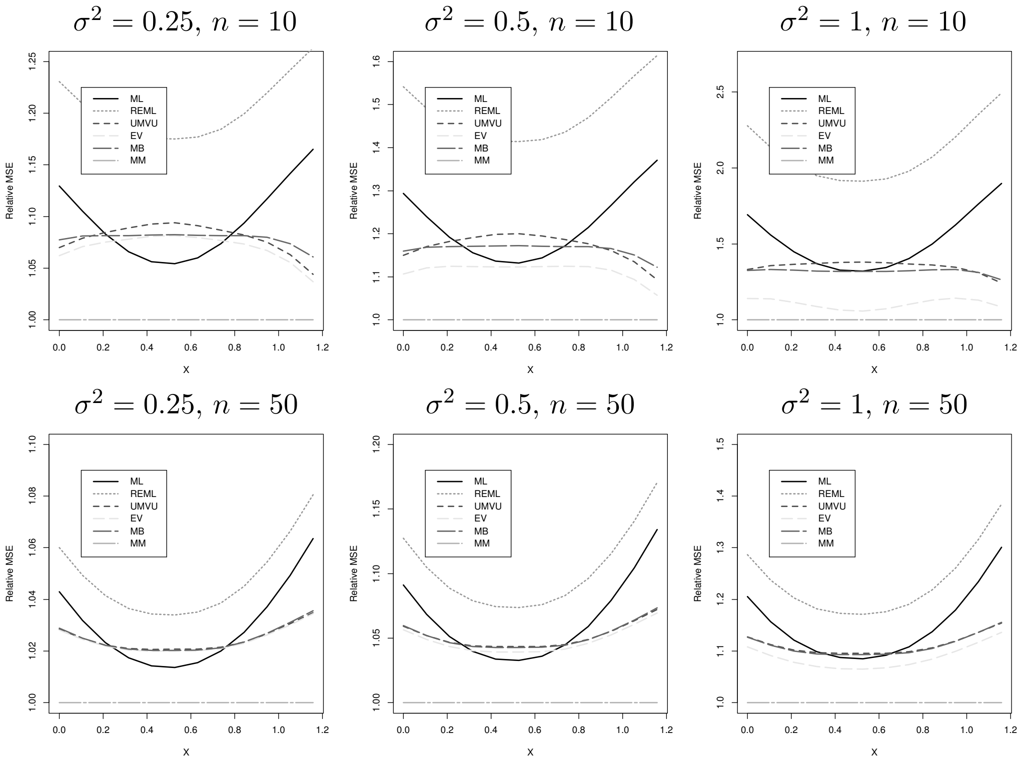

Figure 3. Relative absolute bias for different estimators, which is obtained by dividing the absolute bias by $\mu(x_0)$. The MB/SMB estimators perform the best in most cases.

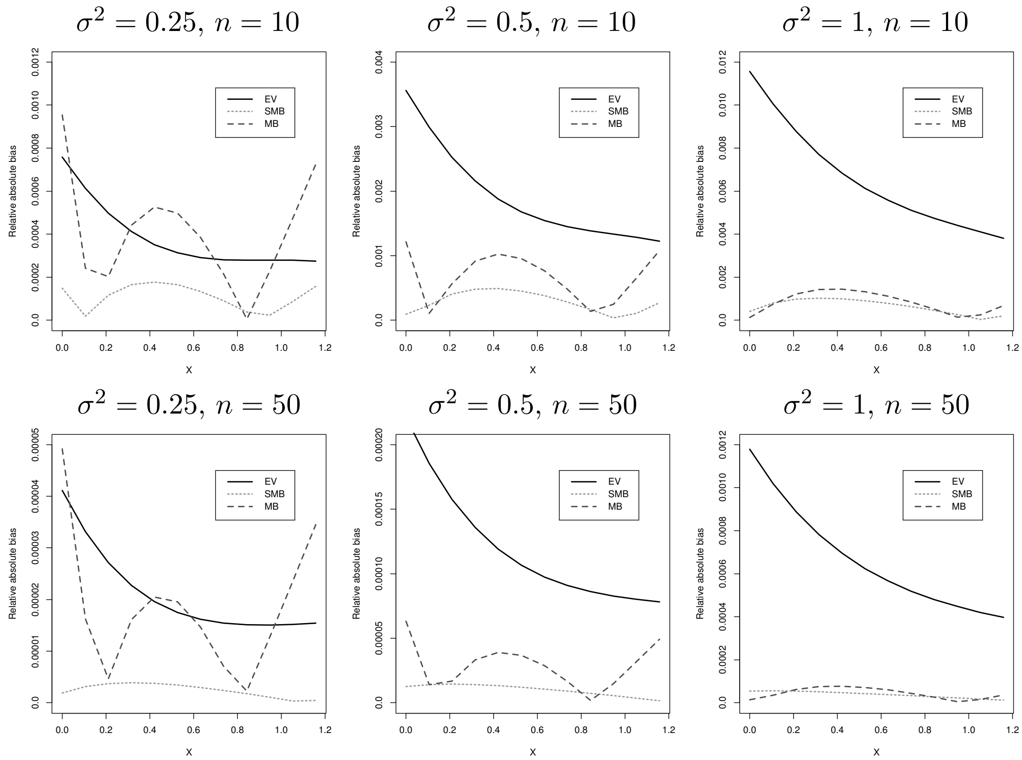

Figure 4. Coverage probabilities for the bootstrap and variance confidence intervals of the MM and MB estimators. The bootstrap intervals have nice coverage properties, and perform better than the variance intervals.

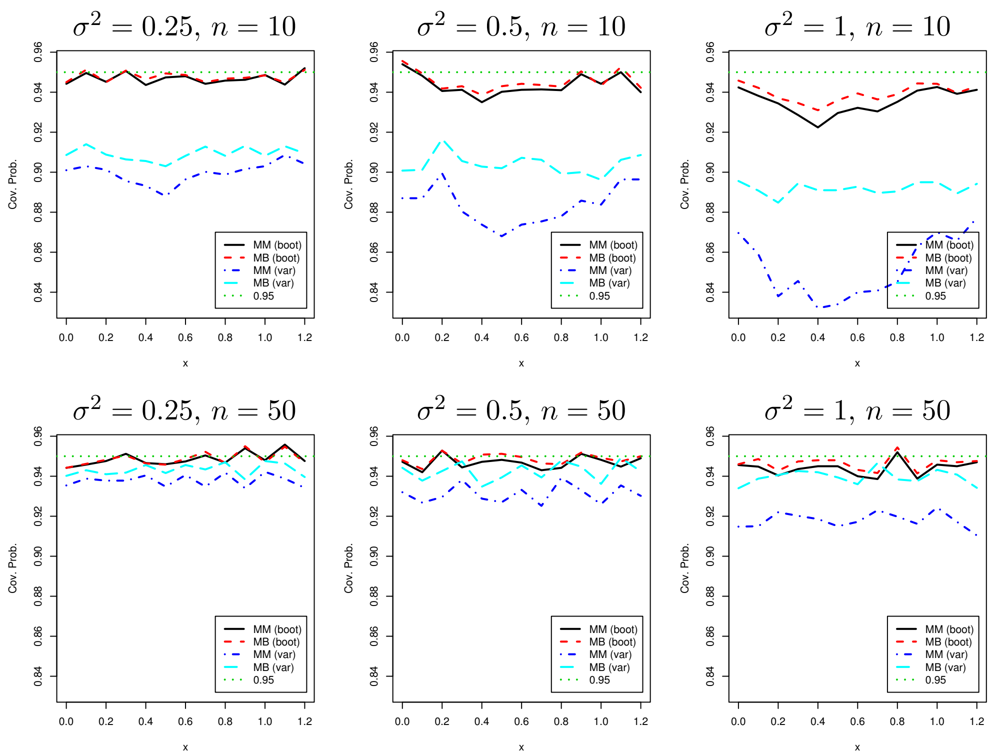

Figure 5. Estimator comparison for the sediment data

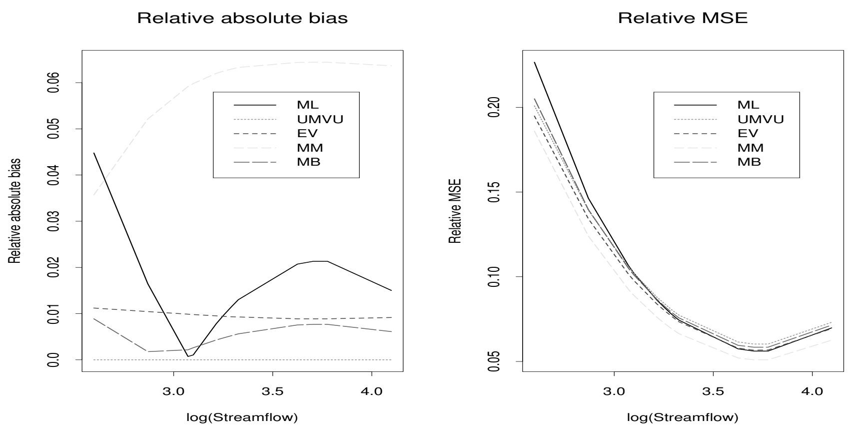

Table 1

Instantaneous values of streamflow (Q) and sediment discharge (L) obtained from a sampling program.

Streamflow (Q) (m^3/s)	Sediment Discharge (L) (metric tons)
125.44	12698.0
70.0	1287.9
68.86	2575.9
27.86	197.7
41.44	1623.5
13.38	25.4
24.92	87.1

Table 2
Daily values of streamflow for a 14-day study period.

Obs	Month	Day	Streamflow (m^3/s)	The MM estimate (metric tons)	95% Boot. CI
1	1	21	13.44	27.34	$[11.45, 105.50]$
2	1	22	17.64	58.46	$[28.89, 187.62]$
3	1	23	40.60	596.72	$[404.43, 1373.96]$
4	1	24	26.88	189.82	$[116.97, 483.91]$
5	1	25	22.18	110.95	$[62.96, 312.30]$
6	1	26	21.59	102.89	$[57.65, 289.72]$
7	1	27	26.88	189.82	$[118.40, 482.12]$
8	1	28	43.68	729.99	$[496.24, 1662.63]$
9	1	29	60.20	1757.64	$[1127.96, 4286.66]$
10	1	30	37.52	479.82	$[325.46, 1099.98]$
11	1	31	27.86	209.77	$[132.42, 523.07]$
12	2	1	25.90	171.13	$[103.91, 443.37]$
13	2	2	24.92	153.65	$[92.34, 405.70]$
14	2	3	24.92	153.65	$[92.34, 405.70]$