Funding a Warranty Reserve with Inputs After Each Sale

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We consider funding an interest-bearing warranty reserve with contributions after each sale. The problem for the manufacturer is to determine the initial level of the reserve fund and the amount to be put in after each sale, so as to ensure that the reserve fund covers all the warranty liabilities with a prespecified probability over a fixed period of time. We assume a non-homogeneous Poisson sales process, random warranty periods, and an exponential failure rate for items under warranty. We derive the mean and variance of the reserve level as a function of time and provide a heuristic to aid the manufacturer in its decision.

1 Overview

Since the Magnuson-Moss Warranty Act of 1975 [20], manufacturers are required to provide a warranty for all consumer goods which cost more than $15. Warranties play an important role in the consumer-manufacturer relationship. They offer assurance to the consumer that their purchase will achieve certain performance standards through at least the warranty period. The manufacturers use warranties as a marketing tool and they limit their liability.

When designing product warranties, the manufacturers must decide on many issues, such as warranty policy, length of warranty period, repair policy, and quality control. They also have to plan to cover the costs associated with the warranty. An issue of critical importance to the manufacturers is managing the costs associated with the warranty effectively. Our research investigates planning for these costs.

We investigate the funding of a warranty reserve account with contributions made after each sale. A warranty reserve is used to accommodate all of the costs associated with the servicing of a warranty of a product. We model a policy that is currently implemented in industry; that of adding a fraction of each sale to the reserve fund. There are a variety of goals that a manufacturer may have regarding its warranty reserve. Two general goals are to keep the reserve above some target dollar amount $B > 0$ and to not have an excessive amount of money in the reserve. The reasoning behind these goals is simple: a shortage requires extra administrative costs and may even have legal ramifications, while an excessive surplus locks money in the reserve that may be more useful for other business interests. Achieving these goals requires careful planning.

2 Literature Review

Warranty theory has been heavily studied over the past two decades. Blischke and Murthy [4] wrote a comprehensive reference for the subject. They discuss many different types of warranty policies, including many warranty policies currently implemented in industry. Numerous cost and optimization models are developed from both the consumer's and the manufacturer's point of view, including life cycle and long-run average cost models. We use these models to compute the expected warranty cost of a product in our numerical examples.

Many of the early papers on warranty theory discuss the costs and other effects that are associated with warranties. Glickman and Berger [8] consider the effect of warranty on demand by assuming that demand increases as the warranty period increases.

Warranty costs affect both the buyer and the seller. Mamer [13] wrote the first paper to provide a comprehensive model of both the buyer's and seller's expected costs and long-run average costs for the free replacement warranty. Our research focuses on the manufacturers' view of warranty costs.

The concept of a warranty reserve is a topic of many research works. The initial papers on warranty reserves discussed here consider a fixed product lot size throughout the life cycle of a product (or equivalently, a fixed cumulative failure rate). Menke [15] wrote one of the first papers to address the warranty reserve problem. He concentrates on calculating the expected warranty cost over a given warranty period for two types of pro-rata warranty policies (linear rebate and lumpsum rebate) assuming a constant product failure rate. Amato and Anderson [1] extend Menke's model by allowing the reserve fund to accrue interest, requiring the consideration of discounted costs. A comparison to Menke's results is made, concluding that discounting significantly reduces the expected warranty reserve over longer periods of time. Both models are rather limited in scope because they only consider pro-rata warranty policies and an exponential failure distribution.

Balcer and Sahin [3] derive the moments of the total replacement cost for both the free-replacement and pro-rata warranty policies during the product life cycle. They assume that successive failure times form a renewal process.

Mamer [14] uses renewal theory to model repeated product failures over a life cycle of the product. He incorporates discounting in his model and allows for a general failure distribution.

Tapiero and Posner [19] allow for a portion of each sale to be set aside for future warranty costs. The contributions to the reserve fund and the items sold occur at a constant rate. The claims are generated according to a compound Poisson Process and they use a sample path technique to compute the long-run probability distribution of the warranty reserve.

Eliashberg, Singpurwalla, and Wilson [7] calculate the reserve for a product whose failure rate is indexed by two scales, time and usage. They allow for a general failure rate and assume a form of imperfect repair. The warranty reserve is computed to minimize a loss function for the manufacturer.

Ja, et al. [10] compute the distribution of the total discounted warranty cost over the life cycle of the product. They analyze the discounted warranty cost of a single sale under many different policies and then consider different stochastic sales processes. A single contribution to the reserve is made at the beginning of the life cycle. However, the subtractions from the reserve due to warranty costs are tracked as a function of time.

Another application related to the warranty reserve problem is the insurance premium problem. An insurance company must decide on the monthly premium to charge a certain class of customer. Low premiums result in loss to the insurer, while high premiums result in loss of business to the competition. A discussion of this can be found in [17]. There are other related problems, including the funding of a company's pension plan. Many of these problems are solved using actuarial models, particularly collective risk (loss) models (see [12] and [5] for references on this subject). However, the current models do not incorporate the number of policies insured by the company at any given time.

The works described above illustrate many different models to compute the warranty reserve. However, they assume that the reserve is either funded at the beginning of the product sales period or at a constant rate. We extend this research by modeling contributions to the reserve after each sale and allowing the cumulative warranty claim rate to depend on the sales process.

3 Notation and Assumptions

We begin by introducing some notation and assumptions. We define $R(t)$ as the amount in the reserve at time t, where $t = 0$ represents the beginning of the period. The reserve fund accrues interest at constant rate $\alpha > 0$. At each sale, an amount c is contributed to the account. The manufacturer must decide on the initial reserve level, R_0, and the contribution amount to the reserve from each sale, c, at the beginning of the period. Let $S(t)$ be the total number of sales up to time t. We assume that $\{S(t), t \geq 0\}$ is a nonhomogeneous Poisson Process with a known rate function $\theta(\cdot)$ (we call this an $NPP\,(\theta(\cdot))$). Each item is under warranty for a random amount of time. The warranty durations are independent and identically distributed with common cdf $F(\cdot)$ and mean w. Also, the warranty durations are independent of any future failures. Note that this allows for a constant warranty period. The customer always makes a warranty claim at each product failure. We assume instantaneous repair and that the repair times of a given item follow a Poisson Process with rate λ. The repair cost of the ith failure (at time Y_i) is D_i, a random variable. The D_i's are i.i.d. and are independent of the failure time. Let $D(t)$ be the total undiscounted cost of all claims up to time t; hence

$$D(t) = \sum_{i:Y_i \leq t} D_i.$$

Let $X(t)$ denote the number of items under warranty at time t and S_j denote the time of the jth sale. The manufacturer observes the number of items under warranty at time 0 to aid in his determination of R_0 and c. The manufacturer may or may not know the remaining warranty lifetimes of the items under warranty at time 0; we consider both cases. Figure 1 illustrates the evolution of the warranty reserve over time.

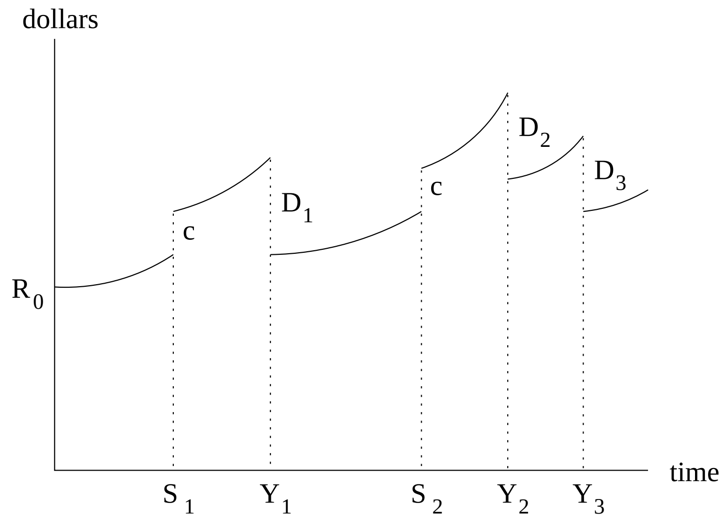

Figure 1: Example of Warranty Reserve Account

For computational purposes, it is helpful to distinguish between the effects of the items sold since time 0 from the items sold before time 0. We will break $X(t)$ into two parts: let $X^n(t)$ represent the number of items under warranty at time t that were sold after time 0, and let $X^o(t)$ represent the number of items under warranty at time t that were sold prior to time 0. We write

$$R(t) = R^n(t) + R^o(t),$$

where $R^n(t)$ is the portion of the reserve related to the new items $X^n(t)$, and $R^o(t)$ is the portion of the reserve related to the old items $X^o(t)$. Thus, in $R^n(t)$, we add contributions from new purchases and only subtract the claims generated by new items. In $R^o(t)$, there are no new contributions, so we only subtract claims generated by old items. Similarly, we define $D^n(t)$ ($D^o(t)$) as the total undiscounted claims from time 0 to t generated by the new (old) items. It is convenient to define $R^n(0) = 0$ and $R^o(0) = R_0$. In our model we track both $R^n(t)$ and $R^o(t)$ for ease in computation, while the manufacturer just tracks $R(t)$.

We will calculate first and second moments for some of the functions $R(t)$, $S(t)$, $X(t)$, $D(t)$ and their components ($R^n(t), R^o(t)$, etc.). We represent this by using lower case for the first moment and using lower case with a subscript of 2 for the second moment (e.g. $r(t) = E[R(t)]$ and $r_2(t) = E[R^2(t)]$). Any exception to this will be mentioned at the appropriate place throughout the thesis. Also, we will use Δ_h to indicate the change in a function from t to $t + h$. For example,

$\Delta_h R(t) = R(t+h) - R(t)$. Finally, we will use the standard $o(h)$ notation for a function $g(h)$ when

$$\lim_{h \to 0} \frac{g(h)}{h} = 0.$$

4 Probability Distribution of the Number of Items Under Warranty

In this section we derive the distributions for $X^n(t)$ and $X^o(t)$.

4.1 Distribution of $X^n(t)$

First we explore the $\{X^n(t), t \geq 0\}$ process. At time t, items are purchased according to an $NPP(\theta(\cdot)))$. The amount of time an item is under warranty is a random variable with cdf $F(\cdot)$. We assume there is no capacity on the total number of items under warranty at any time. Therefore, we can model the $\{X^n(t), t \geq 0\}$ process as an $M_t/G/\infty$ queue with arrival rate $\theta(\cdot)$ and service time distribution $F(\cdot)$.

The following result was established independently by Palm [16] and Khintchine [11]. Most recently, Eick, Massey, and Whitt [6] provided a simpler proof of this result and developed some further results for the $M_t/G/\infty$ queue.

Theorem 1 *Let $Z(t)$ be the number of items in an $M_t/G/\infty$ queue at time t with arrival rate $\theta(\cdot)$ and i.i.d. service times S with cdf $F(\cdot)$. At time t, there are 0 items in the queue. Then, for each time point $t \geq 0$, $Z(t)$ has a Poisson distribution with mean*

$$E\left[\int_{t-S}^{t} \theta(u)du\right] = \int_{0}^{t} \theta(t-u)\left[1 - F(u)\right] du.$$

Therefore, the moments of $X^n(t)$ are

$$x^n(t) = \int_{0}^{t} \theta(t-u)\left[1 - F(u)\right] du, \text{ and} \tag{1}$$

$$x_2^n(t) = x^n(t) + \left(x^n(t)\right)^2. \tag{2}$$

4.2 Distribution of $X^o(t)$

We consider two possible cases for the items sold prior to time 0: either the manufacturer fully knows the remaining warranty durations of all items under warranty at time 0 or that the remaining warranty durations are i.i.d. random variables with common cdf $Q(\cdot)$. The former case is rather easy to handle – the entire sample path of $X^o(t)$ is a deterministic function. If the remaining warranty durations are unknown, the probability that the remaining warranty duration of an item is greater than t, given that it was under warranty at time 0, is $1 - Q(t)$. Hence,

$$X^o(t) \sim Bin\left(X(0), 1 - Q(t)\right),$$

where $X(0)$ is the number of items under warranty at time 0. The moments are

$$x^o(t) = X(0)(1 - Q(t)), \text{ and} \tag{3}$$

$$x_2^o(t) = X(0)Q(t)\left(1 - Q(t)\right) + \left(x^o(t)\right)^2. \tag{4}$$

One choice for $Q(t)$ is obtained from the stationary distribution of the remaining service times in an $M/G/\infty$ queue in steady state. From Takács [18], we have the following lemma.

Lemma 1 (Takács, Theorem 3.2.2) : *Let $X(t)$ be the number of items under warranty at time t, and let $L_i(t)$ denote the remaining warranty period of item i under warranty. The sales process is a Poisson process. If $w < \infty$, we have*

$$\lim_{t\to\infty} P\left(L_i(t) < x_i \;\forall\, i = 1, \ldots, k \,|\, X(t) = k\right) = \prod_{i=1}^{k} \frac{1}{w} \int_0^{x_i} [1 - F(s)]\, ds,$$

and the limiting distribution is independent of the initial state.

Therefore, under the assumption of Poisson input in steady state, we have that the remaining warranty distributions are independent of each other and the probability that an item is still under warranty at time t, given that it was under warranty at time 0 is

$$1 - Q(t) = \frac{1}{w} \int_t^{\infty} [1 - F(s)]\, ds, \tag{5}$$

where $Q(t)$ is determined by Lemma 1, i.e.

$$Q(t) = \frac{1}{w} \int_0^t [1 - F(s)]\, ds. \tag{6}$$

This result can be extended to the case of a non-homogeneous Poisson process, as shown in the following lemma.

Lemma 2 *Let $X(t)$ be the number of items under warranty at time t, and let $L_i(t)$ denote the remaining warranty period of item i under warranty. Suppose that the sales process begins at time $-A$, and $\{S(t), t \geq -A\}$ is a non-homogeneous Poisson process with rate function $\theta(\cdot)$. We have*

$$P\left(L_i(t) < x_i \; \forall\, i = 1, \ldots, k \,|\, X(t) = k\right) = \prod_{i=1}^{k} \frac{\int_0^{t+A} [F(s + x_i) - F(s)]\, \theta(t-s) ds}{\int_0^{t+A} [1 - F(s)]\, \theta(t-s) ds}.$$

Proof. Since the sales process is an $NPP(\theta(\cdot))$, we know that for $t \geq -A$,

$$P(X(t) = k) = \exp\left(- \int_{u=0}^{t+A} (1 - F(u))\theta(t-u) du \right) \frac{\left(\int_{s=0}^{t+A} (1 - F(s))\theta(t-s) ds \right)^k}{k!}. \tag{7}$$

We compute $P\left(L_i(t) < x_i \; \forall\, i = 1, \ldots, k; X(t) = k\right)$. Let $\Theta(u) = \int_{s=-A}^{u} \theta(s) ds$. We have

$$\begin{aligned}
& P\left(L_i(t) < x_i \forall\, i = 1, \ldots, k; X(t) = k\right) \\
&= \sum_{n=k}^{\infty} e^{-\Theta(t)} \frac{\Theta(t)^n}{n!} \binom{n}{k} \left[\frac{1}{\Theta(t)} \int_0^{t+A} F(s)\theta(t-s) ds \right]^{n-k} \\
&\quad \cdot \prod_{i=1}^{k} \left[\frac{1}{\Theta(t)} \int_0^{t+A} [F(x_i + u) - F(u)]\theta(t-u) du \right] \\
&= e^{-\Theta(t)} \frac{\Theta(t)^k}{k!} \sum_{n=k}^{\infty} \frac{1}{(n-k)!} \left[\int_0^{t+A} F(s)\theta(t-s) ds \right]^{n-k} \\
&\quad \cdot \prod_{i=1}^{k} \left[\frac{1}{\Theta(t)} \int_0^{t+A} [F(x_i + u) - F(u)]\theta(t-u) du \right]
\end{aligned}$$

$$= \frac{e^{-\Theta(t)}}{k!} \exp\left(\int_0^{t+A} F(s)\theta(t-s)ds \right) \prod_{i=1}^{k} \int_0^{t+A} [F(x_i+u) - F(u)]\theta(t-u)du$$

$$= \exp\left(- \int_0^{t+A} (1-F(s))\theta(t-s)ds \right) \frac{1}{k!} \prod_{i=1}^{k} \int_0^{t+A} [F(x_i+u) - F(u)]\theta(t-u)du. \tag{8}$$

The conditional probability $P(L_i(t) < x_i \; \forall\, i = 1, \ldots, k \,|\, X(t) = k)$ is Equation (8) divided by Equation (7). We get

$$\prod_{i=1}^{k} \frac{\int_0^{t+A} [F(x_i+u) - F(u)]\theta(t-u)du}{\int_0^{t+A} (1-F(s))\theta(t-s)ds}.$$

This completes the proof. ■

The above lemma implies that for an $NPP(\theta(\cdot))$ sales process, we can use the following for $Q(t)$:

$$Q(t) = \frac{\int_0^{\infty} (F(t+u) - F(u))\, \theta(-u)du}{\int_0^{\infty} (1-F(s))\, \theta(-s)ds}. \tag{9}$$

It is easy to check that Equation (9) reduces to Equation (6) if $\theta(t) = \theta$ for all values of t.

In the next section, we will need the moments of $X(t), X^n(t)$, and $X^o(t)$ in the computation of the moments of $R(t)$. Since $X^n(t)$ and $X^o(t)$ are independent of each other, we compute the moments of $X(t)$ as:

$$x(t) = x^n(t) + x^o(t), \text{ and} \tag{10}$$

$$x_2(t) = x_2^n(t) + x_2^o(t) + 2x^n(t)x^o(t). \tag{11}$$

5 Differential Equations for Moments of $R(t)$

We will consider two cases: $X^o(t)$ is unknown during $[0, T]$ (here we use the distribution discussed in Section 4.2), and $X^o(t)$ is known in its entirety during $[0, T]$. We cover the former case here and the latter case in Section 9.3. In the results that follow, we will need expressions for $E[\Delta_h S(t)]$ and

$E[\Delta_h D(t)]$, where h is small. Since $\{S(t), t \geq 0\}$ is an $NPP\,(\theta(\cdot))$, we know that

$$E[\Delta_h S(t)] = E\left[\int_{u=t}^{t+h} \theta(u)du\right] = \theta(t)h + o(h).$$

The stochastic process $\{D(t), t \geq 0\}$ is a random sum of random variables. Let $N(t)$ represent the number of claims from time 0 to t. For a given sample path of $\{X(t), t \geq 0\}$, $\{N(t), t \geq 0\}$ is an $NPP\,(\lambda X(\cdot))$. The repair costs are i.i.d. with common mean $E[D]$ and second moment $E[D^2]$. Therefore,

$$\begin{aligned} E[\Delta_h D(t)] = E[\Delta_h N(t)]E[D] &= E\left[\int_{u=t}^{t+h} \lambda X(u)du\right] E[D] \\ &= \lambda E[X(t)h]E[D] + o(h) = \lambda x(t)E[D]h + o(h). \end{aligned}$$

Similarly,

$$\begin{aligned} \text{Var}\,(\Delta_h D(t)) &= E[\Delta_h N(t)]\text{Var}(D) + E^2[D]\text{Var}(\Delta_h N(t)) \\ &= \lambda x(t)h\left[E[D^2] - E^2[D]\right] + \lambda x(t)E^2[D]h + o(h) \\ &= \lambda x(t)E[D^2]h + o(h). \end{aligned}$$

We next introduce notation for item failure rates. Consider an arbitrary item that was sold in $[0, t]$. Let U be its time of sale. Then, U has cdf

$$P(U \leq u) = \frac{\Theta(u)}{\Theta(t)}, \qquad 0 \leq u \leq t,$$

$$\text{where } \Theta(t) = \int_0^t \theta(s)ds.$$

Let W represent the warranty period random variable. Then, the probability that the item is under warranty at time t is $P(U + W > t)$. Given that it is under warranty at time t, the probability that its warranty expires in $[t, t+\delta]$ is given by

$$h^n(t)\delta = \frac{f_{U+W}(t)}{1 - F_{U+W}(t)}\delta + o(\delta). \tag{12}$$

Since the warranty periods are i.i.d. and the sales process is an NPP, we see that the items are independent of each other. Hence, if $X^n(t) = i$, the probability that a single items fails in $[t, t+\delta]$ is $ih^n(t)\delta + o(\delta)$. We do a similar analysis for the items under warranty at time 0. We assume that the remaining lifetimes are unknown but are independent of each other. The probability that an item is still under warranty at time t is $1 - Q(t)$. The probability that its warranty expires in $[t, t+\delta]$ is given by

$$h^o(t)\delta = \frac{-Q'(t)}{1 - Q(t)}\delta + o(\delta). \tag{13}$$

For convenience, we define

$$H^i(t) = \int_{s=0}^{t} h^i(s)ds, \text{ for } i \in n, o.$$

We are now ready to compute the moments of $R(t)$. The appendix contains the proofs of the initial lemmas.

Theorem 2 *Let $r(t) = E[R(t)]$. Then,*

$$\frac{dr(t)}{dt} = \alpha r(t) + c\theta(t) - \lambda E[D]x(t), \tag{14}$$

with initial condition $r(0) = R_0$.

Deriving the differential equations for $E[R^n(t)]$ and $E[R^o(t)]$ is similar to Theorem 2.

Lemma 3 *Let $r^n(t) = E[R^n(t)]$ and $r^o(t) = E[R^o(t)]$. Then,*

$$\frac{dr^n(t)}{dt} = \alpha r^n(t) + c\theta(t) - \lambda E[D]x^n(t),$$
$$\frac{dr^o(t)}{dt} = \alpha r^o(t) - \lambda E[D]x^o(t),$$

with initial conditions $r^n(0) = 0$ and $r^o(0) = R_0$.

To derive the differential equation for the second moment, we first give two lemmas.

Lemma 4 *Let* $v(t) = E[R^o(t)X^o(t)]$*. Then*

$$\frac{dv(t)}{dt} = (\alpha - h^o(t))\, v(t) - \lambda E[D] x_2^o(t), \tag{15}$$

with initial condition $v(0) = X(0)R_0$*, and* $h^o(t)$ *is as in Equation (13).*

Lemma 5 *Let* $u(t) = E[R(t)X(t)]$*. Then*

$$\begin{aligned}\frac{du(t)}{dt} =& (\alpha - h^n(t))\, u(t) + c\theta(t)\,(x(t)+1) - \lambda E[D] x_2(t) + \theta(t) r(t) + \\ & (h^n(t) - h^o(t)) * (r^n(t) x^o(t) + v(t)),\end{aligned} \tag{16}$$

with initial condition $u(0) = X(0)R_0$*, and where* $v(t)$ *satisfies Lemma 4,* $h^n(t)$ *satisfies Equation (12),* $h^o(t)$ *satisfies Equation (13), and* $r^n(t)$ *satisfies Theorem 3.*

We are now ready to provide the differential equation for the second moment of $R(t)$.

Theorem 3 *Let* $r_2(t) = E[R^2(t)]$ *and* $r(t), u(t), v(t),$ *and* $r^n(t)$ *be defined as before. Then*

$$\frac{dr_2(t)}{dt} = 2\alpha r_2(t) + c^2\theta(t) + \lambda E[D^2] x(t) + 2c\theta(t) r(t) - 2\lambda E[D] u(t), \tag{17}$$

where $r_2(0) = R_0^2$*.*

Proof. We proceed as in Lemma 4. We have

$$R(t+h) = e^{\alpha h} R(t) + c\Delta_h S(t) - \Delta_h D(t) + o(h).$$

Squaring both sides and rearranging terms, we get

$$\begin{aligned}R^2(t+h) - R^2(t) =& (e^{2\alpha h} - 1) R^2(t) + c^2 (\Delta_h S(t))^2 + (\Delta_h D(t))^2 + 2ce^{\alpha h} R(t) \Delta_h S(t) - \\ & 2e^{\alpha h} R(t) \Delta_h D(t) - 2c \Delta_h S(t) \Delta_h D(t) + o(h).\end{aligned}$$

Taking expectation, we obtain

$$r_2(t+h) - r_2(t) = (e^{2\alpha h} - 1)r_2(t) + c^2 E[\Delta_h S(t)]^2 + E[\Delta_h D(t)]^2 + 2ce^{\alpha h} E[R(t)\Delta_h S(t)] - 2e^{\alpha h} E[R(t)\Delta_h D(t)] - 2cE[\Delta_h S(t)\Delta_h D(t)] + o(h). \quad (18)$$

We investigate each term of the right hand side of Equation (18) below:

(1) $(e^{2\alpha h} - 1)r_2(t) = (2\alpha h + o(h))r_2(t)$.

(2) $c^2 E[\Delta_h S(t)]^2 = c^2 \left(\theta(t)h + \theta^2(t)h^2\right) + o(h) = c^2\theta(t)h + o(h)$.

(3) The mean and variance of $\Delta_h D(t)$ was computed prior to Lemma 1. We have

$$\begin{aligned} E[\Delta_h D(t)]^2 &= Var(\Delta_h D(t)) + E^2[\Delta_h D(t)] \\ &= \lambda E[D^2]x(t)h + (\lambda E[D]x(t)h)^2 + o(h) \\ &= \lambda E[D^2]x(t)h + o(h). \end{aligned}$$

(4) $R(t)$ is independent of $\Delta_h S(t)$ since future sales do not impact the current reserve level. Therefore, $E[R(t)\Delta_h S(t)] = E[R(t)]E[\Delta_h S(t)] = \theta(t)r(t)h + o(h)$.

(5) We calculate $E[R(t)\Delta_h D(t)]$ by conditioning on $X(t)$:

$$\begin{aligned} &\sum_k E[R(t)\Delta_h D(t)|X(t) = k]P[X(t) = k] \\ &= \sum_k E[R(t)|X(t) = k]E[\Delta_h D(t)|X(t) = k]P[X(t) = k] \\ &= \sum_k E[R(t)|X(t) = k]\lambda k P[X(t) = k]E[D]h + o(h) \\ &= \lambda E[R(t)X(t)]E[D]h + o(h) = \lambda E[D]u(t)h + o(h). \end{aligned}$$

(6) We calculate $E[\Delta_h S(t)\Delta_h D(t)]$ by conditioning on $X(t)$:

$$\begin{aligned} &= \sum_k E[\Delta_h S(t)\Delta_h D(t)|X(t) = k]P[X(t) = k] \\ &= \sum_k E[\Delta_h S(t)|X(t) = k]E[\Delta_h D(t)|X(t) = k]P[X(t) = k] \end{aligned}$$

$$= \sum_k \theta(t)h * \lambda E[D]kP[X(t) = k]h + o(h)$$

$$= \theta(t)\lambda E[D]x(t)h^2 + o(h) = o(h).$$

To complete the proof, we substitute the expressions found in (1)-(6) into Equation (18), divide by h, and take the limit as $h \to 0$. ■

Theorem 3 provides a system of equations for the first and second moments of $R(t)$. We can solve this linear system analytically by solving the equations in the following order: $r(t), r^a(t), v(t), u(t), r^2(t)$. This is because each differential equation only uses functions of t that are either known or previously solved in the system – this is known as a triangular system. We can also use a software package, such as MATLAB, to solve the system numerically. Clearly, we can use the solution of the system to find the variance by applying the formula

$$Var(R(t)) = r_2(t) - r^2(t).$$

In the next section, we present the general solution to this system and some examples for simple warranty distributions.

6 Solution for Moments of $R(t)$

We now provide the solution to the differential equations derived in Section 5. For a complete solution, it is necessary to know the functions $x(t)$, $x_2(t)$, $x^n(t)$, $x^o(t)$, $h^o(t)$, $h^n(t)$, $H^o(t)$, and $H^n(t)$. These expressions, defined in Equations (1) – (13) of Section 4, depend only on the warranty distribution $F(\cdot)$ and the sales rate $\theta(\cdot)$.

Theorem 4 *Let $r(t), r^n(t), v(t), u(t)$, and $r_2(t)$ be defined as in Section 5. Then, we have*

$$r(t) = R_0 e^{\alpha t} + e^{\alpha t} \int_{s=0}^{t} e^{-\alpha s} \left(c\theta(s) - \lambda E[D]x(s)\right) ds, \tag{19}$$

$$r^n(t) = e^{\alpha t} \int_{s=0}^{t} e^{-\alpha s} \left(c\theta(s) - \lambda E[D]x^n(s)\right) ds,$$

$$v(t) = X(0)R_0 e^{\alpha t - H^o(t)} - \lambda e^{\alpha t - H^o(t)} E[D] \int\limits_{s=0}^{t} x_2^o(s) e^{-\alpha s + H^o(s)} ds,$$

$$u(t) = X(0)R_0 e^{\alpha t - H^n(t)} + e^{\alpha t - H^n(t)} \int\limits_{s=0}^{t} e^{-\alpha s + H^n(s)} [(h^n(s) - h^o(s)) (r^n(s) x^o(s) + v(s))$$

$$+ c\theta(s)(x(s) + 1) - \lambda x_2(s) E[D] + \theta(s) r(s)] \, ds,$$

$$r_2(t) = R_0^2 e^{2\alpha t} + e^{2\alpha t} \int\limits_{s=0}^{t} e^{-2\alpha s} \left[c^2 \theta(s) + \lambda x_2(s) E[D^2] + 2c\theta(s) r(s) - 2\lambda E[D] u(s)\right] ds.$$

Proof. We apply the techniques to solve linear differential equations for each of $r(t)$, $r^n(t)$, $v(t)$, $u(t)$, and $r_2(t)$. For the sake of brevity, we omit the details. ■

6.1 Example: Constant Warranty Period and Constant Sales Rate

We provide the solution for the first and second moments of $R(t)$ for the example of a constant warranty period w and a constant sales rate function $\theta(t) = \theta$ for all $t \geq 0$. The second moment $r_2(t)$ is quite complex, so we instead provide the variance of $R(t)$.

First, we provide the moments of $X^n(t)$ and $X^o(t)$, and the expressions for $F(t)$, $h^n(t)$, and $h^o(t)$. We assume that the remaining warranty periods of the items sold prior to time 0 is unknown. We apply the result from Section 4.2 to determine the distribution of $X^n(t)$. We have

$$F(t) = \begin{cases} 0, & 0 \leq t < w \\ 1, & t \geq w \end{cases}, \text{ and}$$

$$\int\limits_{0}^{t} \theta\,(t-u)\,[1 - F(u)]\, du = \theta \min(t, w).$$

Applying the formulas for the first and second moments of $X^n(t)$ and $X^o(t)$ yields

$$x^n(t) = \theta \min(t, w),$$

$$x_2^n(t) = \theta \min(t, w) + \theta^2 \min(t^2, w^2),$$

$$x^o(t) = \begin{cases} X(0)\left(\frac{w-t}{w}\right) & 0 \leq t < w \\ 0 & t \geq w \end{cases},$$

$$x_2^o(t) = \begin{cases} X(0)\left(\frac{(w-t)t}{w^2}\right) + \left(X(0)\left(\frac{w-t}{w}\right)\right)^2 & 0 \le t < w \\ 0 & t \ge w \end{cases}.$$

Using the expressions for $h^n(t)$ and $h^o(t)$ defined in (12) and (13), we obtain

$$h^n(t) = \begin{cases} 0, & t < w \\ \frac{1}{w} & t \ge w \end{cases}, \text{ and}$$

$$h^o(t) = \begin{cases} \frac{1}{w-t}, & t < w \\ 0 & t \ge w \end{cases}.$$

We consider two cases.

Case 1. $0 \le t \le w$. Here, we have

$$r(t) = A_0 + A_1 t + A_2 e^{\alpha t}, \text{ and}$$
$$Var(R(t)) = C_0 + C_1 t + C_2 t^2 + C_3 e^{\alpha t} + C_4 t e^{\alpha t} + C_5 e^{2\alpha t},$$

where

$$A_0 = \frac{1}{\alpha^2}\left(\lambda E[D]\theta + \lambda E[D]X(0)\alpha - c\theta\alpha - \frac{\lambda E[D]X(0)}{w}\right),$$
$$A_1 = \frac{\lambda E[D]}{\alpha w}\left(\theta w - X(0)\right),$$
$$A_2 = -\frac{1}{\alpha^2}\left(\lambda E[D]\theta + \lambda E[D]X(0)\alpha - \frac{\lambda E[D]X(0)}{w} - c\theta\alpha - R_0\alpha^2\right),$$
$$C_0 = \frac{1}{4\alpha^4 w^2}\left(6\lambda^2\theta E^2[D]\alpha w^2 - 4\lambda^2 X(0)E^2[D] - 4\lambda c\theta E[D]\alpha^2 w^2 + 6\lambda^2 X(0)E^2[D]\alpha w\right.$$
$$\left.-\lambda\theta E[D^2]\alpha^2 w^2 + \lambda X(0)E[D^2]\alpha^2 w - 2\lambda X(0)E[D^2]\alpha^3 w^2 - 2c^2\theta\alpha^3 w^2\right),$$
$$C_1 = \frac{1}{2\alpha^3 w^2}\left(\lambda X(0)E[D^2]\alpha^2 w - 4\lambda^2 X(0)E^2[D] + 2\lambda^2\theta E^2[D]\alpha w^2\right.$$
$$\left.+2\lambda^2 X(0)E^2[D]\alpha w - \lambda\theta E[D^2]\alpha^2 w^2\right),$$
$$C_2 = -\frac{\lambda^2 X(0)E^2[D]}{\alpha^2 w^2},$$
$$C_3 = \frac{2E[D]}{\alpha^4 w^2}\left(\lambda c\theta\alpha^2 w^2 + \lambda^2 X(0)E[D] - \lambda^2\theta E[D]\alpha w^2 - \lambda^2 X(0)E[D]\alpha w\right),$$
$$C_4 = \frac{2\lambda^2 X(0)E^2[D]}{\alpha^3 w^2}, \text{ and}$$

$$C_5 = \frac{1}{4\alpha^4 w^2}\left(\lambda\theta E[D^2]\alpha^2 w^2 + 2c^2\theta\alpha^3 w^2 + 2\lambda X(0)E[D^2]\alpha^3 w^2 + 2\lambda^2\theta E^2[D]\alpha w^2\right.$$
$$\left. -\lambda X(0)E[D^2]\alpha^2 w - 4\lambda^2 X(0)E^2[D] + 2\lambda^2 X(0)E^2[D]\alpha w - 4\lambda c\theta E[D]\alpha^2 w^2\right).$$

Case 2. $t > w$. In this case, there are no longer any old items under warranty. Therefore, we have

$$r(t) = A_0 + A_1(t-w) + A_2 e^{\alpha(t-w)} + r(w), \text{ and}$$
$$Var(R(t)) = C_0 + C_1(t-w) + C_2 e^{\alpha(t-w)} + C_3 e^{2\alpha(t-w)} + Var(R(w)),$$

where

$$A_0 = \frac{\theta}{\alpha^2}\left(\lambda E[D] - c\alpha\right),$$
$$A_1 = \frac{\theta\lambda E[D]\alpha}{\alpha^2},$$
$$A_2 = \frac{1}{\alpha^2}\left(c\theta\alpha + R_0\alpha^2 - \lambda\theta E[D]\right),$$
$$C_0 = \frac{1}{4\alpha^3}\left(6\lambda^2\theta E^2[D] - 4\lambda c\theta E[D]\alpha - 2c^2\theta\alpha^2 - \lambda\theta E[D^2]\alpha\right),$$
$$C_1 = \frac{1}{2\alpha^2}\left(2\lambda^2\theta E^2[D] - \lambda\theta E[D^2]\alpha\right),$$
$$C_2 = \frac{2}{\alpha^3}\left(\lambda c\theta E[D]\alpha - \lambda^2\theta E^2[D]\right), \text{ and}$$
$$C_3 = \frac{1}{4\alpha^3}\left(\lambda\theta E[D^2]\alpha + 2c^2\theta\alpha^2 - 4\lambda c\theta E[D]\alpha + 2\lambda^2 E[D^2]\theta\right).$$

7 Deciding the Values of c and R_0

The manufacturer must decide on the values of c and R_0 at the beginning of a new period, with the goal of satisfying all of the warranty costs in the period while remaining above some target $B > 0$ with some prespecified probability $1 - \beta$. Of course, setting artificially high values of c and R_0 will achieve this goal, but this will tie up excess money that may be used for other business interests. In this section, we suggest criteria that the manufacturer may use as a basis for its decision.

7.1 Distribution of $R(t)$

We first point out that the variance of $R(t)$ is independent of the initial reserve level, R_0. Let S_i be the time of the ith sale, and let Y_j be the time of the jth failure (with repair cost D_i). Then, the reserve at time t can be computed as:

$$R(t) = R_0 e^{\alpha t} + c \sum_{S_i \le t} e^{\alpha(t-S_i)} - \sum_{Y_i \le t} D_i e^{\alpha(t-Y_i)}. \tag{20}$$

Note that the last two terms of Equation (20) are the random components of $R(t)$ and do not contain R_0. Therefore, the variance is independent of R_0 and only depends on the variable c. We use this fact in deriving a heuristic to determine the values of c and R_0 in the next section.

In general, the distribution of $R(t)$ is difficult to determine. However, if the sales process is a non-homogeneous Poisson Process, the distribution is asymptotic Normal as the number of sales becomes large. A proof of this can be obtained as a minor extension of the result given by Ja [9], which is based on [10]. Therefore, we use the Normal distribution as an approximation. In our simulations, the values of $R(t)$ seem to follow a Normal distribution, especially for large t. We show an example in the next section.

Let $\Phi(\cdot)$ be the standard Normal cumulative distribution function (zero mean and variance one) and let z_β be the number such that $\Phi(z_\beta) = 1 - \beta$. Therefore, at a given point of t, approximately $100(1-\beta)\%$ of sample paths of $R(t)$ will remain above $r(t) - z_\beta\sqrt{Var(R(t))}$. Figure 2 shows two examples of sample paths of $R(t)$ and the $100(1-2\beta)\%$ confidence bands at each value of t. In each graph, the jagged line represents a typical sample path of $R(t)$. The smooth central line is the plot of $r(t)$, while the outer lines are the plots of $r(t) \pm z_\beta\sqrt{Var(R(t))}$. In the left graph, $\min_{t\in[0,T]} r(t) - z_\beta\sqrt{Var(R(t))}$ occurs at time T, while the minimum in the right graph occurs within $(0, T)$.

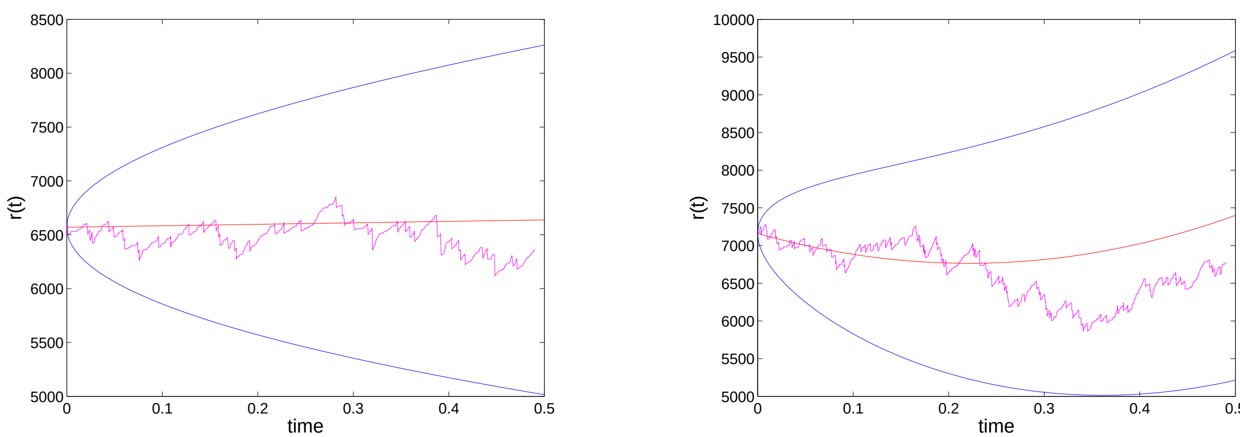

Figure 2: Examples of Confidence Bands for $R(t)$

7.2 Heuristic for Deciding c and R_0

The manufacturer has great flexibility is choosing the values of c and R_0. The original problem is to satisfy all of the warranty claims up to time T and remain above a target B with a given probability. That is, we wish to choose c and R_0 so that

$$\psi(c, R_0) = 1 - P\left(\min_{t \in [0,T]} R(t) \geq B\right) \tag{21}$$

is bounded above by a prespecificed probability. This problem of calculating a ruin probability is very complicated (see [2]). Since we cannot evaluate Equation (21) exactly, we develop an approximation using the result that the distribution of $R(t)$ is approximately Normal. That is,

$$\frac{R(t) - r(t)}{\sqrt{Var(R(t))}} \approx N(0,1),$$

as the number of sales increases to infinity. From this we see that

$$r(t) - z_\beta \sqrt{Var(R(t))} \geq B \Longrightarrow P(R(t) \geq B) \geq 1 - \beta. \tag{22}$$

Now suppose that values c and R_0 are chosen to satisfy

$$\min_{t \in [0,T]} \left(r(t) - z_\beta \sqrt{Var(R(t))}\right) = B. \tag{23}$$

Then, from (22) we see that this choice of (c, R_0) implies that

$$\min_{t \in [0,T]} P(R(t) \geq B) = 1 - \beta.$$

However, the ruin probability $\psi(c, R_0)$ is greater than β, since

$$\psi(c, R_0) = 1 - P\left(\min_{t \in [0,T]} R(t) \geq B\right) \geq 1 - \min_{t \in [0,T]} P(R(t) \geq B) = \beta.$$

Thus β provides a lower bound on the ruin probability $\psi(c, R_0)$. Intuitively, the quantities β and $\psi(c, R_0)$ appear to be related. We use simulation to help uncover a possible relationship between the two quantities.

Therefore, we estimate a parameter q_β so that

$$\min_{t\in[0,T]} \left(r(t) - q_\beta \sqrt{Var(R(t))} \right) = B \tag{24}$$

$$\implies P\left(\min_{t\in[0,T]} R(t) > B \right) \approx 1 - \beta.$$

Clearly we cannot guarantee that such a q_β will work in all possible situations, but we believe such an estimate will be instructive to a manufacturer.

Since there are many values of c and R_0 which will satisfy Equation (23), we offer a heuristic to select one set. The heuristic assumes that we have an additional condition to satisfy: at time T, the expected reserve level is $R_0 e^{\alpha T}$ (to account for accumulated interest). Therefore, we choose c so that

$$r(T) = R_0 e^{\alpha T}.$$

From Equation (19), we see that this is equivalent to solving the equation

$$e^{\alpha T} \int_{s=0}^{T} e^{-\alpha s} \left(c\theta(s) - \lambda E[D] x(s) \right) ds = 0. \tag{25}$$

This equation does not contain R_0. Rearranging Equation (25) to isolate the variable c yields

$$c = \frac{\lambda E[D] \int_{s=0}^{T} e^{-\alpha s} x(s) ds}{\int_{s=0}^{T} e^{-\alpha s} \theta(s) ds}. \tag{26}$$

We use this value of c in solving Equation (23). Since the left hand side of Equation (23) is a monotone increasing function of R_0, there is a unique value of R_0 that satisfies the equation.

We used simulation to estimate q_β for different values of β. For ten different sets of parameter values and distributions, we ran 5000 replications and recorded the minimum value of $R(t)$ (and the occurence time t^*) over $[0, T]$ for each replication. We first recorded the time, t_B, when $r(t) - z_\beta \sqrt{Var(R(t))}$ reached its minimum value over $[0, T]$ for the given parameters. Then, for each simulated replication, we recorded the minimum value of the reserve level $R(t^*)$ and the time of occurence t^*. We selected q so that the simulated reserve process $R(t)$ lies above $r(t) - q\sqrt{Var(R(t))}$

for all $t \in [0, T]$. The quantity q is given by the following formula:

$$q = \frac{r(t_B) - R(t^*)}{\sqrt{Var(R(t_B))}}.$$

We then computed the $100(1-\beta)$th percentile of q for each parameter set i (call this $q_{i\beta}$). Our suggested value of q_β for each value of β is $\max_i q_{i\beta}$. In our experience, the worst cases (large $q_{i\beta}$) occurred when the minimum of $r(t) - z_\beta\sqrt{Var(R(t))}$ did not occur at time T. However, the dispersion in $q_{i\beta}$ for the different parameter sets was not large enough to consider different cases. For reference, we give the values of z_β to compare with our suggested values of q_β. We also give the range and standard deviation of $q_{i\beta}$ to note the relative error in the estimate. The results are summarized in Table 1.

β	z_β	q_β	$\max_i q_{i\beta} - \min_i q_{i\beta}$	$\sqrt{Var(q_{i\beta})}$
.10	1.282	1.842	0.441	0.146
.05	1.645	2.197	0.414	0.152
.025	1.960	2.594	0.501	0.168
.01	2.326	3.059	0.492	0.174
.005	2.576	3.349	0.591	0.205
.001	3.090	4.163	0.814	0.275

Table 1: Suggested Values for q_β

The values for $q_{i\beta}$ in each of the simulation trials are not vastly different from z_β. For $\beta > .01$, the value of $\max q_{i\beta} - \min q_{i\beta}$ was less than 0.5 and the standard deviation of $q_{i\beta}$ was less than 0.2. From our experience, the value of $q_{i\beta}$ is most affected by the variation in the repair costs and the initial number of items under warranty. While these estimates for q_β will not work for all problems, we observed that they were fairly robust for our simulations. We applied the heuristic with these values of q_β for other parameter sets and always had $\{R(t) > B \;\forall\, t\}$ in at least $100(1-\beta)\%$ of the trials.

This heuristic has many advantages: (1) it is easy to compute, (2) provides stability to the expected reserve level from period to period, (3) yields a unique answer, and (4) performs well under simulation.

8 Numerical Computations

We dedicate this section to a numerical example. Consider a non-renewable, free-replacement warranty with constant period one year. All the items are independent and identical, with mean 0.1 failures per year and each replacement cost to the manufacturer is fixed at \$100. The sales process is a Poisson process with mean 1000 items/year. The interest rate on the account is 6% compounded continuously, and we consider a period T of one-half year. Some products that might have this structure are electronic devices (such as calculators) or small appliances (such as toasters or microwaves). More complex products, such as computers, have similar properties where repairs are "good as new".

Let $E[C_W(\alpha)]$ be the expected total warranty cost discounted to present value for a single item. One option for the manufacturer is to contribute this amount to the reserve after each sale. We compute $E[C_W(\alpha)]$ by the following formula (see Section 4.2 of [4]):

$$E[C_W(\alpha)] = E[D] \int_0^W e^{\alpha t} dM(t), \tag{27}$$

where $M(\cdot)$ is the ordinary renewal function associated with the product failure distribution (here $M(t) = \lambda t$). Substituting the numbers of the example yields $E[C_W(\alpha)] = 9.71$. We use this as a comparison for the recommended value of c obtained through the heuristic.

First, we illustrate the effect of $X(0)$ on the mean of the reserve. Figure 3 plots $r(t)$ for $X(0) = \{500, 1000, 1500, 2000\}$. Note that the expected number of items under warranty in steady state is given by $\theta w = 1000$. In each case, we set the reserve contribution from each sale to $E[C_W(\alpha)] = 9.71$.

This plot clearly shows that letting $c = E[C_W(\alpha)]$ is very effective if $X(0) \approx \theta w$. However, if these two values are far apart, another value of c is recommended. In the instance when $X(0) = 2000$, it requires a value of $c = 17.51$ to keep the expected reserve level at the end of the period equal to $R_0 e^{\alpha T}$. Conversely, a value of $c = 6.24$ achieves the same goal when $X(0) = 500$.

Next, we illustrate the heuristic to determine c and R_0 when there are 1500 items under warranty at time 0 and the remaining warranty periods of these items are unknown. Suppose that the manufacturer must keep the reserve level above $B = 5000$ for the entire period $[0, T]$ with 95%

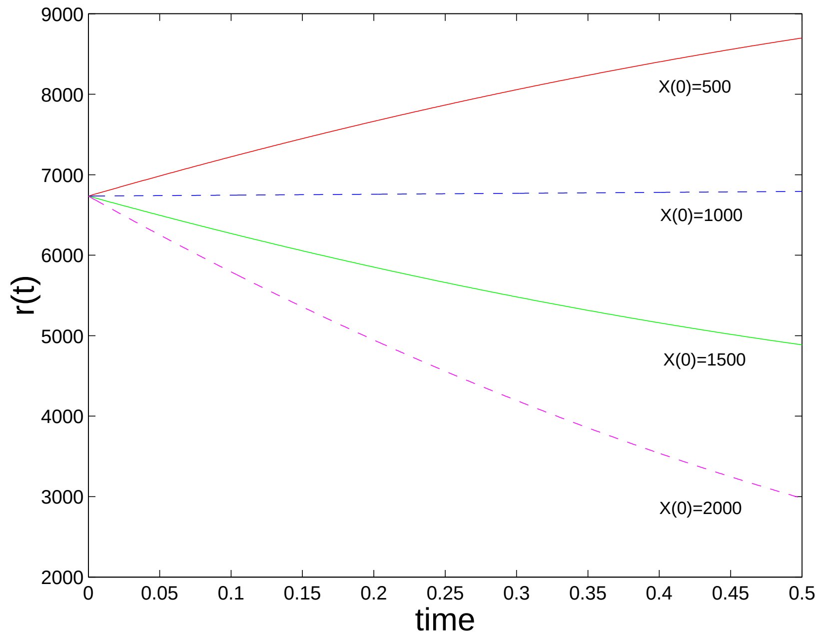

Figure 3: Expected Reserve for Various Values of $X(0)$

probability. Plugging the parameters into Equation (26), we get $c = 13.756$. We see from Table 1 that $q_{.05} = 2.197$. We solve Equation (24) for R_0, obtaining a value of $R_0 = 6734.8$.

We simulated 5000 replications with the above parameters to illustrate the effectiveness of the heuristic and check for normality of $R(t)$. For reference, this set of parameters is different from the ten parameter sets used to determine q. Of the 5000 replications, 223 (4.46%) of them fell below the target $B = 5000$ at some point during the period $[0, T]$. This is slightly less than the target of 5%. We recorded the values of $R(t)$ of each replication at time points $t = 0.125, 0.25, 0.375$, and 0.5. We compare the average and standard deviation of the simulated replications with the theoretical values calculated in Section 6. Also, we give the p-value of the chi-squared test for a Normal distribution. We summarize the results in Table 2.

		Time t			
		0.125	0.25	0.375	0.5
Mean	Sim.	6658.2	6662.9	6757.9	6928.9
	Theor.	6668.6	6680.3	6770.5	6939.8
Standard Deviation	Sim.	453.5	641.8	775.9	886.7
	Theor.	454.8	636.7	772.1	882.9
p-value of χ^2-test		$1.24 \cdot 10^{-7}$	0.137	0.259	0.720

Table 2: Simulation Results

As expected, the simulated mean and standard deviation at each time point match up with the theoretical mean and variance. Clearly, the p-value of the χ^2-test increases with increasing t. Although normality is clearly violated at $t = 0.125$, it is accepted at $t \geq 0.25$. Since the distribution is asymptotic Normal with respect to increasing sales, we can be fairly confident that a large sales process will have an approximately Normal distribution. Our example has a modest sales rate of 1000 items/yr; we see more intensive sales processes lead to a Normal distribution much quicker.

9 Warranty Reserve: Extensions

In this chapter we consider three separate extensions to our basic warranty reserve model:

- The reserve contribution after the jth sale is C_j, a random variable. (Section 9.1)
- There are multiple products for the manufacturer which use the same reserve fund. For each product i, the sales rate is $\theta_i(\cdot)$, the warranty period has cdf $F_i(\cdot)$, and the reserve contribution after each sale of product i is a constant c_i. (Section 9.2)
- The entire sample path of $X^o(t)$ is known during $[0, T]$. This case arises if the manufacturer has access to the time of sale and the length of the warranty period of each item under warranty. (Section 9.3)

9.1 Random contribution to the reserve after each sale

Suppose that the contribution to the reserve after the ith sale is a random variable. We assume that the successive contributions $\{C_i, i \geq 1\}$ are i.i.d. and independent of the reserve level and the sales process. A possible reason for having a random contribution amount independent of the other relevant stochastic processes is when the product costs a different amount in different distribution areas of the manufacturer. This assumption is more critical in application areas such as insurance models or pension funds. We can compute the moments of $R(t)$ under this new assumption as in Section 5. Not surprisingly, the only effect on the system of differential equations is a change from c to $E[C]$ and from c^2 to $E[C^2]$.

Theorem 5 *Let the reserve contributions after each sale be i.i.d. random variables with common mean $E[C]$ and second moment $E[C^2]$. Suppose they are independent of the reserve level and the* *sales process. Then,*

$$\begin{aligned}
\frac{dr(t)}{dt} &= \alpha r(t) + E[C]\theta(t) - \lambda E[D]x(t),\\
\frac{dr^n(t)}{dt} &= \alpha r^n(t) + E[C]\theta(t) - \lambda E[D]x^n(t),\\
\frac{dv(t)}{dt} &= (\alpha - h^o(t))\, v(t) - \lambda E[D]x_2^o(t),\\
\frac{du(t)}{dt} &= (\alpha - h^n(t))\, u(t) + E[C]\theta(t)\,(x(t)+1) - \lambda E[D]x_2(t) + \theta(t)r(t)\\
&\quad + (h^n(t) - h^o(t)) * (r^n(t)x^o(t) + v(t)),\\
\frac{dr_2(t)}{dt} &= 2\alpha r_2(t) + E[C^2]\theta(t) + \lambda E[D^2]x(t) + 2E[C]\theta(t)r(t) - 2\lambda E[D]u(t).
\end{aligned}$$

Proof. We recalculate each term in the derivations of the equations where the term c originally occurred. We use the assumption that the contribution is independent of the reserve level and the sales process to write $E[C_iM(t)] = E[C]E[M(t)]$ and $E[C_i^2M(t)] = E[C^2]E[M(t)]$, where $M(t)$ is a stochastic process independent of C_i. This occurs twice in the derivations of $\frac{dr(t)}{dt}$, $\frac{dr^n(t)}{dt}$, and $\frac{du(t)}{dt}$, and three times in $\frac{dr_2(t)}{dt}$. ■

9.2 Multiple products using a single reserve

Suppose that a manufacturer manages warranties for k products which may be k different products, the same product with k different warranty policies, or a combination of the two. It may be desirable for the manufacturer to have a single warranty reserve for all of these products. We can apply our previous results to handle this situation. For each product i, we assume that the sales process is an $NPP(\theta_i(\cdot))$, the failure rate is exponential with rate λ_i, the contribution to the reserve after each sale is c_i, and each of these are independent of the other products. We track the total reserve contribution from each of the products and denote this as $R_i(t)$. The mean and variance of $R_i(t)$ is computed from the results in Section 5.

Since the products are independent of each other, the total expected reserve $E[R(t)]$ can be computed as a sum of the independent expected reserves of each product and the total variance $Var(R(t))$ is the sum of the individual variances of each product:

$$E[R(t)] = E\left[\sum_{i=1}^{k} R_i(t)\right] = \sum_{i=1}^{k} E[R_i(t)],$$

$$\sqrt{Var(R(t))} = \sqrt{\sum_{i=1}^{k} Var(R_i(t))}.$$

It is clearly beneficial to maintain a combined account rather than k separate ones, due to pooling of the risks involved. Mathematically, this is because the variances combine additively, so the standard deviation of the combined reserve is less than the sum of the standard deviation of k individual reserves.

9.3 $X^o(t)$ is known during $[0, T]$

The case where $X^o(t)$ is known is much easier than when it is unknown. We still divide $X(t)$ as

$$X(t) = X^o(t) + X^n(t),$$

with the change that $X^o(t)$ is a deterministic quantity.

Theorem 6 *Let* $r(t) = E[R(t)]$, $r^n(t) = E[R^n(t)]$, $y(t) = E[R^n(t)X^n(t)]$, *and* $r_2(t) = E[R^2(t)]$. *Assume that* $\{X^o(t), 0 \le t \le T\}$ *is known. Then*

$$\begin{aligned}
\frac{dr(t)}{dt} &= \alpha r(t) + c\theta(t) - \lambda E[D]x(t), \\
\frac{dr^n(t)}{dt} &= \alpha r^n(t) + c\theta(t) - \lambda E[D]x^n(t), \\
\frac{dy(t)}{dt} &= (\alpha - h^n(t))\, y(t) + c\theta(t)\,(1 + x^n(t)) + \theta(t)r^n(t) - \lambda E[D]x_2^n(t), &(28)\\
\frac{dr_2(t)}{dt} &= 2\alpha r_2(t) + c^2\theta(t) + \lambda E[D^2]x(t) + 2c\theta(t)r(t) \\
&\quad - 2\lambda E[D]\,[X^o(t)r(t) + r^o(t)x^n(t) + y(t)]\,, &(29)
\end{aligned}$$

where $r(0) = R_0$, $r^n(0) = 0$, $u(0) = 0$, *and* $r_2(0) = R_0^2$.

The proof is contained in the appendix.

A manufacturer might use both assumptions regarding $X^o(t)$. For example, if they obtain a new computer package that tracks every customer's warranty expiration date during $[0, T]$, they may use the original assumption in the period $[0, T]$ but use their additional information in the period $[T, 2T]$. We leave as future work the case where the purchase times of the items sold before time 0 are known, but the remaining warranty periods are not.

10 Conclusions and Future Work

In this paper, we addressed the problem of modeling a warranty reserve where contributions are made after each sale (where the sales process is a nonhomogeneous Poisson process). We derived the mean and variance of the reserve for all time points t and offered a heuristic for determining good values of c, the contribution amount after each sale, and R_0, the initial reserve level. We also provided extensions of the model by allowing a random contribution amount and multiple products from the same manufacturer.

There are many extensions of our basic warranty reserve model for future research. One simple extension is to allow a fluctuating interest rate. We mention several other extensions below:

- **Bulk arrivals:** The sales occur in batches with a common probability distribution. The successive batch arrivals are independent of each other, but the warranty periods within each batch may be dependent.

- **Ruin Probability:** A common problem in risk theory is to compute the probability of ruin prior to time t. That is, we want to analytically compute the probability, in terms of the problem parameters, that the reserve account drops below a given amount $B > 0$ prior to time t. Asmussen [2] provides an extensive reference on ruin probabilities. However, the specific case of premiums depending on the number of items or policies is not addressed.

- **Reserve-dependent Contributions:** We allow the reserve contribution to depend on the reserve level and the number of items under warranty. There are two possible approaches: to determine a contribution function $c(R(t), X(t))$ prior to time 0 that can be applied for the entire fiscal period, or to dynamically determine the contribution amount at time t.

- **Criteria for Selecting c and R_0:** In this dissertation, we selected c and R_0 so the reserve level remains above a target B with approximate $100(1-\beta)\%$ probability. We can consider other objectives for the manufacturer, such as minimizing a penalty function. That is, if the reserve falls below the target B, the manufacturer incurs sanctions as a function of the time below the target and the amount below the target.

- **Other Applications:** We believe adaptations of our reserve management model can be applied to pension funds and insurance accounts. Currently, the literature on these subjects considers a constant premium input and a static number of customers. For pension funds, we would track the workers currently eligible for pension and the workers eligible for pension in the future. The contribution amount can be a function of both of these, while the subtraction amount will depend on the former. The insurance model would consider many classes of customers (as our model allows), but may have different failure distributions.

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Proof of Theorem 2:

Proof. We look at the change in the reserve from time t to time $t+h$, where h is small. We have

$$R(t+h) - R(t) = (e^{\alpha h} - 1)R(t) + c\,[\Delta_h S(t)] - [\Delta_h D(t)] + o(h).$$

Taking expectation on both sides, we get

$$\begin{aligned} r(t+h) - r(t) &= (e^{\alpha h} - 1)r(t) + c\,(E[\Delta_h S(t)]) - (E[\Delta_h D(t)]) + o(h), \\ &= (\alpha h + o(h))r(t) + c(\theta(t)h + o(h)) - (\lambda E[D]x(t)h + o(h)) \end{aligned}$$

Diving by h and taking the limit as $h \to 0$ yields Equation (14). ■

Proof of Lemma 4:

Proof. We again look at $v(t+h) - v(t)$ and take limits as $h \to 0$.

$$\begin{aligned} v(t+h) &= E[R^o(t+h)X^o(t+h)] \\ &= E[\Big(e^{\alpha h}R^o(t) - \Delta_h D^o(t)\Big)\,(X^o(t) + \Delta_h X^o(t)) + o(h)], \\ v(t+h) - v(t) &= e^{\alpha h}E[R^o(t)\Delta_h X^o(t)] + (e^{\alpha h} - 1)E[R^o(t)X^o(t)] - E[\Delta_h D^o(t)\Delta_h X^o(t)] - \\ &\quad E[\Delta_h D^o(t)X^o(t)] + o(h), \\ v(t+h) - v(t) &= (1 + \alpha h + o(h))\,E[R^o(t)\Delta_h X^o(t)] + (\alpha h + o(h))\,v(t) - \\ &\quad E[\Delta_h D^o(t)\Delta_h X^o(t)] - E[\Delta_h D^o(t)X^o(t)] + o(h). \qquad (30) \end{aligned}$$

We can find $E[R^o(t)\Delta_h X^o(t)]$, $E[\Delta_h D^o(t)X^o(t)]$, and $E[\Delta_h D^o(t)\Delta_h X^o(t)]$ by conditioning on $X^o(t)$.

$$\begin{aligned} E[R^o(t)\Delta_h X^o(t)] &= \sum_i E[R^o(t)\Delta_h X^o(t)|X^o(t) = i]P[X^o(t) = i] \\ &= \sum_i E[R^o(t)|X^o(t) = i]E[\Delta_h X^o(t)|X^o(t) = i]P[X^o(t) = i] \\ &= \sum_i -E[R^o(t)|X^o(t) = i]ih^o(t)P[X^o(t) = i]h + o(h) \\ &= -h^o(t)\sum_i E[R^o(t)i|X^o(t) = i]P[X^o(t) = i]h + o(h) \end{aligned}$$

$$= -h^o(t)v(t)h + o(h).$$

$$\begin{aligned}
E[\Delta_h D^o(t)X^o(t)] &= \sum_i E[\Delta_h D^o(t)X^o(t)|X^o(t) = i]P[X^o(t) = i] \\
&= \sum_i iE[\Delta_h D^o(t)|X^o(t) = i]P[X^o(t) = i] \\
&= \sum_i i^2\lambda P[X^o(t) = i]E[D]h + o(h) \\
&= \lambda E[D]x_2^o(t)h + o(h).
\end{aligned}$$

$$\begin{aligned}
E[\Delta_h D^o(t)\Delta_h X^o(t)] &= \sum_i E[\Delta_h D^o(t)\Delta_h X^o(t)|X^o(t) = i]P[X^o(t) = i] \\
&= \sum_i E[\Delta_h D^o(t)|X^o(t) = i]E[\Delta_h X^o(t)|X^o(t) = i]P[X^o(t) = i] \\
&= \sum_i -i\lambda i h^o(t)E[D]P[X^o(t) = i]h^2 + o(h) \\
&= -\lambda E[D]x_2^o(t)h^o(t)h^2 + o(h) = o(h).
\end{aligned}$$

Plugging these three expressions into Equation (30), dividing by h, and taking the limit as $h \to 0$ completes the result. ■

Proof of Lemma 5:

Proof. We proceed as in Lemma 4. We have

$$\begin{aligned}
u(t+h) &= E\left[R(t+h)X(t+h)\right] \\
&= E\left[\left(e^{\alpha h}R(t) + c\Delta_h S(t) - \Delta_h D(t)\right)(X(t) + \Delta_h X(t)) + o(h)\right], \\
u(t+h) - u(t) &= E[(e^{\alpha h} - 1)R(t)X(t)] + cE[\Delta_h S(t)X(t)] - E[\Delta_h D(t)X(t)] + \\
&\quad E[e^{\alpha h}R(t)\Delta_h X(t)] + cE[\Delta_h S(t)\Delta_h X(t)] - E[\Delta_h D(t)\Delta_h X(t)] + o(h). \qquad (31)
\end{aligned}$$

We investigate each term on the right hand side of Equation (31) below:

(1) $E[(e^{\alpha h} - 1)R(t)X(t)] = (\alpha h + o(h))u(t)$.

(2) $cE[\Delta_h S(t)X(t)] = cE[\Delta_h S(t)]E[X(t)] = c\theta(t)x(t)h + o(h)$ (The number of additional sales from t to $t+h$ is independent of the number of items under warranty at time t).

(3) We calculate $E[\Delta_h D(t)X(t)]$ by conditioning on $X(t)$:

$$\sum_k E[\Delta_h D(t)X(t)|X(t)=k]P[X(t)=k] = \sum_k kE[\Delta_h D(t)|X(t)=k]P[X(t)=k]$$
$$= \sum_k \lambda k^2 P[X(t)=k]E[D]h + o(h) = \lambda E[D]x_2(t)h + o(h).$$

(4) $E[R(t)\Delta_h X(t)] = E[R(t)\Delta_h X^o(t)] + E[R(t)\Delta_h X^n(t)]$.

We calculate $E[R(t)\Delta_h X^o(t)]$ by conditioning on $X^o(t)$:

$$\begin{aligned}
E[R(t)\Delta_h X^o(t)] &= \sum_i E[R(t)\Delta_h X^o(t)|X^o(t)=i]P[X^o(t)=i] \\
&= \sum_i E[R(t)|X^o(t)=i]E[\Delta_h X^o(t)|X^o(t)=i]P[X^o(t)=i] \\
&= \sum_i -ih^o(t)E[R(t)|X^o(t)=i]P[X^o(t)=i]h + o(h) \\
&= -h^o(t)\sum_i E[R(t)i|X^o(t)=i]P[X^o(t)=i]h + o(h) \\
&= -h^o(t)E[R(t)X^o(t)]h + o(h) \\
&= -h^o(t)E\left[\left(R^n(t)+R^o(t)\right)X^o(t)\right]h + o(h) \\
&= -h^o(t)\left(E[R^n(t)X^o(t)] + E[R^o(t)X^o(t)]\right)h + o(h) \\
&= -h^o(t)\left(r^n(t)x^o(t) + v(t)\right)h + o(h).
\end{aligned}$$

We calculate $E[R(t)\Delta_h X^n(t)]$ by conditioning on $X^n(t)$:

$$\begin{aligned}
E[R(t)\Delta_h X^n(t)] &= \sum_i E[R(t)\Delta_h X^n(t)|X^n(t)=i]P[X^n(t)=i] \\
&= \sum_i E[R(t)|X^n(t)=i]E[\Delta_h X^n(t)|X^n(t)=i]P[X^n(t)=i] \\
&= \sum_i E[R(t)|X^n(t)=i]\left(\theta(t)h - ih^n(t)h\right)P[X^n(t)=i] + o(h) \\
&= \theta(t)h\sum_i E[R(t)|X^n(t)=i]P[X^n(t)=i] \\
&\quad - h^n(t)h\sum_i E[R(t)i|X^n(t)=i]P[X^n(t)=i] + o(h)
\end{aligned}$$

$$= \theta(t)r(t)h - h^n(t)E[R(t)X^n(t)]h + o(h)$$
$$= \theta(t)r(t)h - h^n(t)\left(E[R(t)X(t)] - E[R(t)X^o(t)]\right)h + o(h)$$
$$= \theta(t)r(t)h - h^n(t)\left(u(t) - r^n(t)x^o(t) - v(t)\right)h + o(h)$$
$$= \theta(t)r(t)h - h^n(t)u(t)h + h^n(t)\left(r^n(t)x^o(t) + v(t)\right)h + o(h).$$

(5) To calculate $E[\Delta_h S(t)\Delta_h X(t)]$, we must consider the dependence of $S(t)$ and $X(t)$. If there is a sale in $\Delta_h t$, then both $\Delta_h S(t)$ and $\Delta_h X(t)$ are 1. This happens with probability $\theta(t)h + o(h)$. If there is an expiration, then $\Delta_h X(t)$ is -1 while $\Delta_h S(t)$ is 0 (hence their product is 0). Therefore,

$$cE[\Delta_h S(t)\Delta_h X(t)] = c\theta(t)h + o(h).$$

(6) We calculate $E[\Delta_h D(t)\Delta_h X(t)]$ by conditioning on $\Delta_h X(t)$:

$$\sum_k E[\Delta_h D(t)\Delta_h X(t)|\Delta_h X(t) = k]P[\Delta_h X(t) = k]$$
$$= \sum_k kE[\Delta_h D(t)|\Delta_h X(t) = k]P[\Delta_h X(t) = k]$$
$$= \sum_k \frac{\lambda k^2 E[D]}{2} P[\Delta_h X(t) = k]h + o(h)$$
$$= \frac{\lambda E[D]}{2} h(\theta^2(t)h^2 + \theta(t)h) + o(h) = o(h).$$

To complete the proof, we substitute the expressions found in (1)-(6) into Equation (31), divide by h, and take the limit as $h \to 0$. ■

Proof of Theorem 6:

Proof. We apply the same proof technique as in Section 5. Most of the calculations are exactly the same; we will mention and omit these and just provide the changes.

The calculations for $\frac{dr(t)}{dt}$ and $\frac{dr^n(t)}{dt}$ are the same as in Theorems 2 and 3. Since $\frac{dy(t)}{dt}$ is a new quantity, we provide the calculations.

$$y(t+h) = E\left[R^n(t+h)X^n(t+h)\right]$$
$$= E\left[\left(e^{\alpha h}R^n(t) + c\Delta_h S(t) - \Delta_h D(t)\right)\left(X^n(t) + \Delta_h X^n(t)\right) + o(h)\right]$$

$$y(t+h) - y(t) = E[(e^{\alpha h} - 1)R^n(t)X^n(t)] + cE[\Delta_h S(t)X^n(t)] - E[\Delta_h D^n(t)X^n(t)] + \\ E[e^{\alpha h}R^n(t)\Delta_h X^n(t)] + cE[\Delta_h S(t)\Delta_h X^n(t)] - E[\Delta_h D^n(t)\Delta_h X^n(t)] + o(h) \tag{32}$$

We investigate each term on the right hand side of Equation (32). The expressions for $E[(e^{\alpha h} - 1)R^n(t)X^n(t)]$, $cE[\Delta_h S(t)X^n(t)]$, $E[\Delta_h S(t)\Delta_h X^n(t)]$, and $E[\Delta_h D^n(t)\Delta_h X^n(t)]$ are computed very similarly to their counterparts from Lemma 5. The computations yield

$$\begin{aligned} E[(e^{\alpha h} - 1)R^n(t)X^n(t)] &= (\alpha h + o(h))y(t), \\ cE[\Delta_h S(t)X^n(t)] &= c\theta(t)x^n(t)h + o(h), \\ E[\Delta_h S(t)\Delta_h X^n(t)] &= c\theta(t)h + o(h), \text{ and} \\ E[\Delta_h D^n(t)\Delta_h X^n(t)] &= o(h) \end{aligned}$$

We compute the other two quantities below:

(1) We calculate $E[\Delta_h D^n(t)X^n(t)]$ by conditioning on $X^n(t)$:
$\sum_k E[\Delta_h D^n(t)X^n(t)|X^n(t) = k]P[X^n(t) = k] = \sum_k kE[\Delta_h D^n(t)|X^n(t) = k]P[X^n(t) = k]$
$= \sum_k \lambda k^2 P[X^n(t) = k]E[D]h + o(h) = \lambda E[D]x_2^n(t)h + o(h).$

(2) We calculate $E[R^n(t)\Delta_h X^n(t)]$ by conditioning on $X^n(t)$:

$$\begin{aligned} E[R^n(t)\Delta_h X^n(t)] &= \sum_i E[R^n(t)\Delta_h X^n(t)|X^n(t) = i]P[X^n(t) = i] \\ &= \sum_i E[R^n(t)|X^n(t) = i]E[\Delta_h X^n(t)|X^n(t) = i]P[X^n(t) = i] \\ &= \theta(t)r^n(t)h - \sum_i ih^n(t)E[R^n(t)|X^n(t) = i]P[X^n(t) = i] + o(h) \\ &= \theta(t)r^n(t)h - h^n(t)E[R^n(t)X^n(t)]h + o(h) \\ &= \theta(t)r^n(t)h - h^n(t)y(t)h + o(h). \end{aligned}$$

To obtain Equation (28), we substitute the expressions found in (1)-(6) into Equation (32), divide by h, and take the limit as $h \to 0$.

It remains to derive Equation (29). We proceed in the usual fashion, beginning with Equation (18) derived in Section 5. The only change from the derivation in Theorem 3 is in $E[R(t)\Delta_h D(t)]$; we provide that below:

$$\begin{aligned}
E[R(t)\Delta_h D(t)] &= \sum_k E[R(t)\Delta_h D(t)|X(t)=k]P[X(t)=k] \\
&= \sum_k E[R(t)|X(t)=k]E[\Delta_h D(t)|X(t)=k]P[X(t)=k] \\
&= \sum_k E[R(t)|X(t)=k]\lambda k P[X(t)=k]E[D]h + o(h) \\
&= \lambda E[R(t)X(t)]E[D]h + o(h) \\
&= \lambda E[D]\left[X^o(t)r(t) + r^o(t)x^n(t) + E[R^n(t)X^n(t)]\right] + o(h) \\
&= \lambda E[D]\left[X^o(t)r(t) + r^o(t)x^n(t) + y(t)]\right] h + o(h).
\end{aligned}$$

Making this substitution, we obtain Equation (29), completing the proof. ∎