

## COMPREHENSIVE WRITTEN EXAM – STOR655 MATHEMATICAL STATISTICS

Unless otherwise noted, all problem parts have equal weight. In budgeting your time expect that some part will take longer than others. When solving multi-part problems feel free to use results of earlier parts even if you cannot solve them in proving later parts. Do not forget to split the time between both papers.

1. Let  $f_n(x)$  and  $f(x)$  be densities such that  $f_n(x) \rightarrow f(x)$  a.e. Prove or disprove  $\int |f_n(x) - f(x)| dx \rightarrow 0$ .
2. Prove or disprove: Let  $b_n \rightarrow \infty$ ,  $b_n(X_n - a) \xrightarrow{\mathcal{D}} X$  and  $g$  be a twice differentiable function satisfying  $g'(a) = 0$ . Then

$$b_n^2(g(X_n) - g(a)) \xrightarrow{\mathcal{D}} \frac{g''(a)}{2} X^2.$$

3. Let us consider  $X_1, \dots, X_n$  i.i.d. with density

$$f(x|\theta) = \begin{cases} \frac{2x}{\theta} & \text{when } 0 < x \leq \theta, \\ \frac{2(1-x)}{1-\theta} & \text{when } \theta < x < 1, \\ 0 & \text{otherwise,} \end{cases}$$

where the parameter  $\theta \in [0, 1]$ .

- (a) Is this an exponential family?
- (b) Let  $0 < \theta_1 < \theta_2 < 1$ . Evaluate  $KL(f(x|\theta_1), f(x|\theta_2))$  and  $KL(f(x|\theta_2), f(x|\theta_1))$
- (c) Find the method of moments estimator of  $\theta$ .
- (d) Is the MM estimator consistent? Is it asymptotically normal? What is its asymptotic variance?
- (e) Find the log likelihood  $l(\theta|\mathbf{x})$  and describe how would you find the maximum likelihood estimator.
- (f) **This is worth 2x.** Is that MLE is strongly consistent?
- (g) **Optional.** Is MLE asymptotically normal? What is its asymptotic variance? Compare to MM.