CWE 2016: STOR 654 Questions

Note: All parts (7 of them) have an equal weight although their degrees of difficulty vary.

[1] Consider a 3D random vector (X, Y, Z) with mean vector μ and covariance matrix

$$\Sigma = \begin{pmatrix} 1 & \rho & 0 \\ \rho & 1 & \rho \\ 0 & \rho & 1 \end{pmatrix}.$$

(1a) Explain what is wrong in the following contradicting statements. When $\rho = -1$, X and Y are colinear that implies a degenerate joint distribution of (X, Y, Z), but Σ is nonsingular.

(1b) For what values of ρ a trivariate normal distribution of (X, Y, Z) can be defined over $\mathbb{R}^3$?

(1c) Suppose a normal density for (X, Y, Z) is expressed as $f(x, y, z) = c_0\, e^{-Q/2}$ under the condition in $(1b)$ with the quadratic form
$Q = c_1(x^2 + y^2 + z^2) - (xy + xz + yz)$, where c_0, c_1 are positive constants. Specify the values of ρ, c_0 and c_1.

(1d) Show that the density $f(x, y, z)$ in $(1c)$ would be supported over a subspace of dimension lower than 3 when $\rho = -1/\sqrt{2}$. Specify the dimensionality of its support.

[2] **Useful facts:** If $U \sim beta(a, b)$ — the beta distributions with parameters $a > 0$ and $b > 0$, then U has the density

$$\pi(u) = \frac{u^{a-1}(1-u)^{b-1}}{B(a,b)}, \quad u \in [0, 1],$$

where $B(a, b) = \int_0^1 v^{a-1}(1-v)^{b-1}\, dv$, and $EU = \frac{a}{a+b}$.

Assume X and Y are independent random variables with $X \sim Bernoulli(\lambda)$ and $Y \sim Bernoulli(\mu)$.

(2a) Let λ and μ be iid random variables with the common uniform distribution over the interval $(0,1)$. Denote the prior (joint) density for (λ, μ) by π. Consider estimation of the 2D parameter (λ, μ) under the squared error loss $\|(T_1, T_2) - (\lambda, \mu)\|^2$ where $\| \ \|$ is the Euclidean distance in $\mathbb{R}^2$. Find a Bayesian estimator under the prior π with components T_1 and T_2 as functions of (X, Y). [Hint: Notice that $\|(T_1, T_2) - (\lambda, \mu)\|^2 = |T_1 - \lambda|^2 + |T_2 - \mu|^2$ and use some simple property of π.]

(2b) Explain why it does not make sense to consider Bayesian tests for testing $H_0 : \lambda = \mu$ vs $H_1 : \lambda \neq \mu$ with the prior π given in (2a).

(2c) Given $\beta \in (0,1)$, define a new (mixture) prior $p = \beta\pi_0 + (1-\beta)\pi_1$: on the set $\Theta_0 = \{\lambda = \mu\}$, the component of p is a 1D uniform density π_0 over the interval $(0,1)$; on the set $\Theta_1 = \{\lambda \neq \mu\}$, the component of p is a 2D density π_1 — same as π restricted to Θ_1. Find the Bayesian test for H_0 vs H_1 under the prior p and based on the observation (x, y).