STOR 635 **Exam: CWE** **Year:** 2015

There are 4 questions. Attempt all questions. You may appeal to any result proved in class without proof unless you are specifically asked to "**Give a complete proof.** ". State any result you use. All questions are worth the same number of total points (10 points). Points for parts of a question can be found in boxes on the right. *Even if you don't know the complete solution DONT STRESS OUT. Put down your attempt.* Partial credit will be assigned so try your best on each question.

1. Fix a probability mass function $\mathbf{p} := \{p_k\}_{k\geqslant 0}$ and let $\{Z_n\}_{n\geqslant 0}$ with $Z_0 = 1$ be a branching process with offspring distribution $\mathbf{p}$. More precisely, let $\{\xi_{i,j} : i, j \geqslant 1\}$ be **i.i.d** with distribution $\mathbf{p}$. Now define the sequence $\{Z_n\}_{n\geqslant 0}$ recursively with $Z_0 = 1$ and let

$$Z_n := \sum_{j=1}^{Z_{n-1}} \xi_{n,j} \qquad n \geqslant 1,$$

with the understanding that if $Z_{n-1} = 0$ then $Z_n = 0$. Thus $\xi_{n,j}$ is interpreted as the number of children of individual j in generation $n-1$. Define the probability generating function of $\mathbf{p}$ as

$$\phi(s) := \sum_{k=0}^{\infty} s^k p_k, \qquad s \in [0,1]$$

Suppose there exists a unique $0 < \rho < 1$ such that $\phi(\rho) = \rho$.

(a) Show that the sequence $\left\{\rho^{Z_n} : n \geqslant 0\right\}$ is a Martingale. Here the filtration $\{\mathcal{F}_n : n \geqslant 0\}$ is defined by $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and for $n \geqslant 1$, $\mathcal{F}_n = \sigma(\{\xi_{mj} : 1 \leqslant m \leqslant n, j \geqslant 1\})$. **Give a complete proof.**

[5]

(b) Give reasons why there should be a limit random variable Y such that

$$\rho^{Z_n} \xrightarrow{a.s.} Y.$$

[3]

(c) Calculate $\mathbb{E}(Y)$. Give reasons for your answer (don't just put down the answer). [2]

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2. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\{\mathcal{F}_n : n \geqslant 1\}$ be a collection of increasing sub σ-fields namely $\mathcal{F}_n \subseteq \mathcal{F}$ for all $n \geqslant 1$ and

$$\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \mathcal{F}_3 \subseteq \cdots$$

Think of $\mathcal{F}_n$ as the amount of information on day n. Define the sigma field

$$\mathcal{F}_\infty := \sigma(\cup_{n=1}^{\infty} \mathcal{F}_n).$$

Show that for any $A \in \mathcal{F}_\infty$, and any $\varepsilon > 0$, you can find an $n < \infty$ and $B \in \mathcal{F}_n$ (both n and B typically depend on A and ε) such that

$$\mathbb{P}(A \Delta B) \leqslant \varepsilon.$$

Here $B\Delta A := (B \setminus A) \cup (A \setminus B)$.

Hint: Good set principle

Point of the problem: You are showing that you can **approximate** any set in $\mathcal{F}_\infty$ by sets that you know about in "finite" time.

[10]

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3. Let $\{X_n : n \geqslant 1\}$ be a collection of independent Bernoulli random variables. More precisely assume there exists a sequence $\{p_n : n \geqslant 1\}$ with $0 < p_n < 1$ such that

$$\mathbb{P}(X_n = 1) = p_n, \qquad \mathbb{P}(X_n = 0) = 1 - p_n.$$

Assume that as $n \to \infty$,

$$p_n \to 0, \qquad \text{but} \sum_{j=1}^{n} p_j(1 - p_j) \to \infty.$$

Let $S_n = \sum_{j=1}^{n} X_j$. Show that

$$\frac{S_n - \sum_{j=1}^{n} p_j}{\sqrt{\sum_{j=1}^{n} p_j(1 - p_j)}} \xrightarrow{d} Z,$$

where $Z = N(0, 1)$.

[10]

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4. Suppose $\{X_n : n \geqslant 1\}$ and $\{Z_n : n \geqslant 1\}$ are two collections of random variables. You know the following about these random variables.

   (i) $\{X_n : n \geqslant 1\}$ are centered ($\mathbb{E}(X_n) = 0$ for all n), uniformly bounded second moments (there exists a constant $C < \infty$ such that $\mathbb{E}(X_n^2) < C$ for all n) and are uncorrelated (for all $i \neq j$ $\mathbb{E}(X_i X_j) = 0$). Think of X_n as re-centered stock price changes on day n. **Note:** The $\{X_n : n \geqslant 1\}$ need **not** be independent **or** identically distributed.

   (ii) The collection $\{Z_n : n \geqslant 1\}$ are large shocks to the system namely for each n

$$\mathbb{P}(Z_n = -3n^3) = \frac{1}{2n^2}, \qquad \mathbb{P}(Z_n = 2n^3) = \frac{1}{2n^2}, \qquad \mathbb{P}(Z_n = 0) = 1 - \frac{1}{n^2}.$$

   Again these $\{Z_n : n \geqslant 1\}$ need not be independent and need not be independent of the sequence $\{X_n : n \geqslant 1\}$.

Define $S_n = \sum_{j=1}^{n} X_j$ and $S_n' = \sum_{j=1}^{n}(X_j + Z_j)$.

(a) Show that if you did not take into account the shocks namely considered just S_n then

$$\frac{S_n}{n} \xrightarrow{\text{P}} 0.$$

Give a complete proof.

[4]

(b) Show that even if you considered the shocks then you get the same result:

$$\frac{S_n'}{n} \xrightarrow{\text{P}} 0.$$

[6]

STOR 635 End