1. (15 points.) Precisely state and prove the Monotone Convergence Theorem for Conditional Expectations.

2. (20 points.) Let \( \{X_n, \mathcal{F}_n\} \) be a uniformly integrable sub-martingale on some probability space. Show that for any stopping time \( N \), the family \( \{X_n\wedge N, n \geq 1\} \) is uniformly integrable.

3. (15 points.) Let \( \{\mu_n, n \geq 1\}, \mu \) be probability measures on \((\mathbb{R}, \mathcal{B}(\mathbb{R}))\) such that \( \mu_n \) converge weakly to \( \mu \). Show that the sequence \( \{\mu_n, n \geq 1\} \) is tight.

4. (10 + 15 + 5 points.) Let \( \{X_n\} \) be an independent sequence of mean 0, variance 1 random variables. Let \( S_n = X_1 + \cdots + X_n \) and suppose that \( \frac{S_n}{\sqrt{n}} \) converges in distribution to a standard normal random variable.

   (a) Show that for every \( x \in \mathbb{R} \),
   
   \[ \limsup_{n \to \infty} P(S_n > \sqrt{n}x) \leq P(S_{\infty} > \sqrt{n}x, i.o.) \leq P(S > x), \]
   
   where \( S = \limsup_{n \to \infty} \frac{S_n}{\sqrt{n}} \).

   (b) By using Kolmogorov’s 0–1 law show that \( \limsup_{n \to \infty} \frac{S_n}{\sqrt{n}} = +\infty \) a.s. [Hint: Recall that if a r.v. \( U \) is measurable with respect to the tail sigma-field of an independent sequence of random variables then for some \( c \in [-\infty, \infty] \), \( P(U = c) = 1 \).]

   (c) Recall from Skorohod’s representation theorem, one can find random variables \( \tilde{S}_n \) (having same distribution as \( S_n \)) and a standard normal random variable \( Z \) given on a common probability space such that \( \lim_{n \to \infty} \frac{\tilde{S}_n}{\sqrt{n}} = Z \) a.s. Why doesn’t this contradict the statement in part (b) of the problem?

5. (20 points.) Let \( X_n \geq 0 \) be independent for \( n \geq 1 \). Show that the following are equivalent.

   (i) \( \sum_{n=1}^{\infty} X_n < \infty \) a.s.
   (ii) \( \sum_{n=1}^{\infty} [P(X_n > 1) + \mathbb{E}(X_n 1_{X_n \leq 1})] < \infty \)
   (iii) \( \sum_{n=1}^{\infty} \mathbb{E}(\frac{X_n}{1+X_n}) < \infty \).