Problem 1. Let $\mu$ be a finite measure on $(X, S)$ where $S = \sigma(E)$ with a field $E$. Show that, for $A \in S$ and any $\epsilon > 0$, there is $A_\epsilon \in E$ such that $|\mu(A) - \mu(A_\epsilon)| \leq \mu(A \triangle A_\epsilon) < \epsilon$. Show, in particular, that the first inequality in the latter relation holds.

Problem 2. Without providing any proofs, describe the steps in the construction of the Lebesgue-Stieltjes measure on the real line (including the completion step). If the Lebesgue-Stieltjes measure $\mu$ is a point mass at 0, what is $B_\mu$, the $\sigma$-field obtained by completing the $\sigma$-field $B$ of Borel sets under $\mu$?

Problem 3. (a) Provide an example of a nonmeasurable function $f$ on some measurable space $(X, S)$ such that $f^2$ is measurable. (b) If $f$ and $g$ are measurable functions on a measurable space $(X, S)$, show that $f + g$ is also measurable.

Problem 4. Let $X = \mathbb{R}$, $S = B(\mathbb{R})$, $\mu$ = Lebesgue measure. Consider the function
\[ f(x) = \begin{cases} 0, & x < 0 \\ 1/2^k, & 2k \leq x < 2k + 1 \\ -1/3^k, & 2k + 1 \leq x < 2k + 2, \quad k = 0, 1, 2, \ldots. \end{cases} \]
Compute $\int_X f(x)\mu(dx)$ by using the definition of integral. Give an example of another measure $\mu$ on $(\mathbb{R}, B)$ for which this integral is not defined.

Problem 5. Let $\mu$ be the Lebesgue measure on $(\mathbb{R}, B)$ and $\mu_F$ be the Lebesgue-Stieltjes measure on $(\mathbb{R}, B)$ induced by the function $F(x) = x^{2005}$, $x \in \mathbb{R}$. Show that $\mu_F \ll \mu$ and determine the Radon-Nikodym derivative $d\mu_F/d\mu$. Is it true that $\mu_F \sim \mu$?

Problem 6. Give an example of a sequence of measurable functions $f_n$ on some measure space $(X, S, \mu)$ such that (at the same time)
- $f_n$ does not converge $\mu$-a.e. on $X$,
- $f_n$ converges in measure $\mu$, and
- $f_n$ converges in $L^p(X, S, \mu)$ for $p < 2006$ but not for $p > 2006$.

Good luck!