Let \((X_i, Y_i), i = 1, 2, \ldots\) be a sequence of iid 2D random vectors with the following common distribution: \(X_1\) follows a uniform distribution over the interval \((-\theta, \theta)\), and \(Y_1 = X_1^k\), where \(\theta\) is a positive real number and \(k\) is a positive integer. For a positive integer \(n\), we write \((X, Y)^n = \{(X_1, Y_1), \ldots, (X_n, Y_n)\}\), \(X^n = \{X_1, \ldots, X_n\}\), and \(Y^n = \{Y_1, \ldots, Y_n\}\).

(1) Let \(\theta\) be an unknown parameter and \(k = 2\). Show that \(X_1\) and \(Y_1\) are dependent.

(2) Following the assumptions in (1), does it make sense to estimate the correlation \(\rho\) between \(X_1\) and \(Y_1\) based on a joint sample \((X, Y)^n\)? If “yes”, then propose an estimator for \(\rho\); if “no”, explain why.

(3) Assume both \(\theta\) and \(k\) are unknown parameters. Derive the general formula for \(EX_m^n\) for an arbitrary positive integer \(m\), based on which construct estimators for \(\theta\) and \(k\) based on a sample \(Y^n\) using method-of-moments. Make additional assumptions if needed.

(4) Show that the density for \(Y_1\) is given by

\[
f_{Y_1}(y) = \begin{cases} 
\frac{1}{\theta k} y^{\frac{1}{k}-1}, & -\theta^k < y < 0 \text{ or } 0 < y < \theta^k, \quad \text{if } k \text{ is odd;} \\
\frac{1}{\theta k} y^{\frac{1}{k}-1}, & 0 < y < \theta^k, \quad \text{if } k \text{ is even.}
\end{cases}
\]

(5) Show that \((U_n, V_n)\) is a 2D minimal sufficient statistic for \((\theta, k)\) based on a sample \(Y^n\), where

\[
U_n = \max_{1 \leq i \leq n} |Y_i| \quad \text{and} \quad V_n = \prod_{i=1}^{n} |Y_i|.
\]

(6) Define \(\alpha = \theta^k\) and \(\beta = 1/k\). Find a MLE \((\hat{\alpha}, \hat{\beta})\) for \((\alpha, \beta)\) based on \(Y^n\).

(7) Assume \(k = 2\) and \(\theta\) follows a prior uniform distribution over the interval \([1, 2]\). Find the Bayes estimator for \(\theta\) based on \(Y^n\) under the absolute error loss.

(8) Assume \(k = 2\) and \(\theta\) follows a prior uniform distribution over the interval \([1, 2]\). Conduct the Bayesian test based on \(Y^n\) for the hypotheses \(H_0 : \theta \leq 3/2\) vs \(H_1 : \theta > 3/2\).

(9) Show the following consistency result: \(U_n\) given in (5) converges to \(\alpha\) in probability with an exponential rate as \(n \to \infty\).