1. (15 points) Let $\xi_1, \xi_2, \cdots$ be independent with $\mathbb{E}(\xi_m) = 0$ and $\text{var}(\xi_m) = \sigma_m^2 < \infty$. Let $s_n^2 = \sum_{m=1}^{n} \sigma_m^2$ and $S_n = \sum_{m=1}^{n} \xi_m$. Show that $S_n^2 - s_n^2$ is a martingale.

2. (20 points) Suppose $\{X_n^1\}$ and $\{X_n^2\}$ are supermartingales with respect to some filtration $\{\mathcal{F}_n\}$ and $N$ is a stopping time such that $X_N^1 \geq X_N^2$ a.s. Show that

$$Y_n = X_{n1}^11_{N>n} + X_{n2}^21_{N\leq n}$$

is a supermartingale.

3. (15 points)
   a. (5 pts) Let $\mu$ be a probability measure on the infinite product space $((\mathbb{R}^{\infty}, B(\mathbb{R}^{\infty})))$. Say what it means for $\mu$ to be a product measure.
   
   b. (10 pts) Let $\{X_k\}_{k \geq 1}$ be a sequence of $\mathbb{R}$ valued independent random variables, given on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $X = (X_1, X_2, \cdots)$. Let $\mu$ be the probability distribution of $X$. Show that $\mu$ is a product measure.

4. (20 points) Let $0 \leq X_1 \leq X_2 \cdots$ be random variables such that $EX_n = an^\alpha$, with $a, \alpha > 0$, and $\text{var}(X_n) \leq Bn^\beta$ with $\beta < 2\alpha$. Show that $X_n/n^\alpha \to a$ a.s.

5. (15 points) Prove the "conditional" Hölder's inequality: Let $X, Y$ be random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ such that for some $p, q > 1$, $\mathbb{E}|X|^p < \infty$ and $\mathbb{E}|Y|^q < \infty$. Let $\mathcal{G}$ be a sub $\sigma$-field of $\mathcal{F}$. Show that

$$\mathbb{E}(|XY| | \mathcal{G}) \leq [\mathbb{E}(|X|^p | \mathcal{G})]^{1/p} [\mathbb{E}(|Y|^q | \mathcal{G})]^{1/q}, \text{ a.s.}$$

6. (15 points) Let $\mu_n \sim N(a_n, 1)$, where $a_n$ is a real sequence. Show that the sequence $\{\mu_n\}$ is tight iff $a_n$ is a bounded sequence.