STOR 654 Exam

In budgeting your time expect that some parts will take longer than others. When solving multi-part problems feel free to use results of earlier parts even if you cannot solve them in proving later parts.

1. Is it possible for a random vector \((X, Y)\) to have the following properties: \(EX = 6, EY = 5, EX^2 = 42, EY^2 = 29, EXY = 25?\) (If yes, show an example, if no explain.)

2. Let \(0 < \lambda < 1\) and \(X\) be a random variable such that \(EX^2 < \infty\) and \(EX \geq 0\). Prove or disprove

\[
P(X \geq \lambda EX) \geq (1 - \lambda)^2 \frac{(EX)^2}{EX^2}.
\]

(Hint: Consider a random variable \(X' = XI_{\{X \geq \lambda EX\}}\).)

3. Let \(\{X_n\}\) be the sequence of random variables \(X_n = \mu + \sigma_n Z\), where \(\mu \in \mathbb{R}\), \(Z\) is a random variable with mean 0 and variance 1, and \(\{\sigma_n\}\) is a sequence of strictly positive numbers such that \(\sigma_n \to 0\) as \(n \to \infty\). If \(f : \mathbb{R} \to \mathbb{R}\) is a function continuously differentiable in the neighborhood of \(\mu\) and \(f'(\mu) \neq 0\), show that as \(n \to \infty\),

\[
\frac{f(X_n) - f(\mu)}{\sigma_n f'(\mu)} \xrightarrow{P} Z,
\]

4. Define \(M_{\lambda,n} = \max(X_1, \ldots, X_n)\), where \(X_1, \ldots, X_n\) are i.i.d. Exponential(\(\lambda\)) random variables and \(\lambda > 0\).

(a) Find the c.d.f. of \(M_{\lambda,n}\).

(b) Prove or disprove \(M_{\lambda,n} \xrightarrow{P} \infty\) as \(n \to \infty\) (\(\lambda\) is fixed).

(c) Prove or disprove \(M_{\log(n),n} \xrightarrow{P} 1\) in probability as \(n \to \infty\).
(d) Find a sequence \( c_n \) such that \( c_n(M_{\log(n),n} - 1) \overset{D}{\to} S \), where \( S \) is non-degenerate. What is the c.d.f. of \( S \)?

(Hint: The following fact from calculus might be useful:

\[
\left(1 - \frac{1}{n^{1+s/\log n}}\right)^n \to e^{-e^{-s}}.
\]

(e) Prove or disprove that \( \lambda M_{\lambda,n} - c_n \overset{D}{\to} S \) with the same \( c_n \) and \( S \) as in the previous part. (Hint: You can do this part even if you cannot do part 4d.)