Exam consists of nine questions. Total points: 100.

Name:

1. (5 points) Let \( \{X_n\}_{n \geq 1} \) be an i.i.d. sequence such that \( \mathbb{E}(X_1) = 0 \) and \( \text{Var}(X_1) = 1 \). What can you say about \( \mathbb{P}(\sum_{n=1}^{\infty} X_n \text{ converges}) \)? Give detailed reasons.

2. (10 points) Suppose \( X, Y, Z, U \) are square integrable random variables on some probability space such that \( (X, U) \) and \( (Y, U) \) have the same probability law. Suppose that \( Y = \mathbb{E}(Z \mid U) \) a.s. Show that \( X = Y \) a.s.

3. (15 points) Let \( \{X_n\}_{n \geq 0} \) be a sequence of nonnegative integrable random variables adapted to some filtration \( \{\mathcal{F}_n\} \). Let \( \{\tau_k\}_{k \geq 1} \) be a sequence of \( \mathcal{F}_n \) stopping times such that \( \tau_k \uparrow \infty \) as \( k \to \infty \). Suppose that, for each \( k \geq 1 \), \( \{Y_n = X_{n \wedge \tau_k}, n \geq 0\} \) is a martingale. Show that \( \{X_n\}_{n \geq 0} \) is a supermartingale. If additionally, \( \mathbb{E}(X_n) = \mathbb{E}(X_0) \) for each \( n \), then show that \( \{X_n\}_{n \geq 0} \) is a martingale.

4. (10 points) State the Martingale Convergence Theorem. Show by constructing an example that if the main assumption of the theorem is not satisfied then the conclusion of the theorem may fail.

5. (10 points) Give an example of a sequence \( \{X_n\}_{n \geq 1} \) of random variables that is \( L^1 \) bounded but not uniformly integrable.

6. (15 points) Let \( \{X_n\}_{n \geq 0} \) be a martingale with \( X_0 = 0 \) and let \( N \) be a stopping time. State a theorem precisely that gives general conditions under which \( \mathbb{E}(X_N) = 0 \). Let \( \{\xi_n\}_{n \geq 1} \) be i.i.d., mean 0 random variables such that \( |\xi_n| \leq C \), for each \( n \geq 1 \). Suppose \( X_k = \xi_1 + \cdots + \xi_k \), for \( k \geq 1 \) and \( X_0 = 0 \). Let \( N = \inf \{k : X_k > 1\} \). Can \( \mathbb{E}(N) \) be finite? Give reasons.

7. (10 points) Give an example of a sequence of probability measures \( \{\mu_n\} \) on some Polish space \( S \) such that \( \mu_n \Rightarrow \mu \) but for some open subset \( G \) of \( S \), \( \liminf_{n \to \infty} \mu_n(G) > \mu(G) \).

8. (15 points) The following is an incomplete statement of Scheffe’s Theorem: Let \( f_n, f \) be nonnegative measurable functions on \( \mathbb{R} \) such that \( f_n(x) \to f(x) \), as \( n \to \infty \), for each \( x \in \mathbb{R} \). Suppose that \( \int_{\mathbb{R}} f_n(x) \, dx = 1 \) for each \( n \geq 1 \). Then \( f_n \to f \) in \( L^1 \). Show by constructing an example that the above statement is false in general. Give the complete and correct statement of Scheffe’s Theorem.

9. (10 points) Show that an i.i.d. sequence \( \{X_n\}_{n \geq 0} \) of \( \mathbb{R} \) valued random variables is tight.