August, 2009 Name:________________________

STOR 654 Exam

In budgeting your time expect that some parts will take longer than others. When solving multi-part problems feel free to use results of earlier parts even if you cannot solve them in proving later parts.

1. Is it possible for a random vector (X, Y) to have the following properties: $EX = 6, EY = 5, EX^2 = 42, EY^2 = 29, EXY = 25$? (If yes, show an example, if no explain.)

2. Let $0 < \lambda < 1$ and X be a random variable such that $EX^2 < \infty$ and $EX \geq 0$. Prove or disprove

$$P(X \geq \lambda EX) \geq (1 - \lambda)^2 \frac{(EX)^2}{EX^2}.$$

   (Hint: Consider a random variable $X' = XI_{\{X \geq \lambda EX\}}$.)

3. Let $\{X_n\}$ be the sequence of random variables $X_n = \mu + \sigma_n Z$, where $\mu \in \mathbb{R}$, Z is a random variable with mean 0 and variance 1, and $\{\sigma_n\}$ is a sequence of strictly positive numbers such that $\sigma_n \to 0$ as $n \to \infty$. If $f : \mathbb{R} \to \mathbb{R}$ is a function continuously differentiable in the neighborhood of μ and $f'(\mu) \neq 0$, show that as $n \to \infty$,

$$\frac{f(X_n) - f(\mu)}{\sigma_n f'(\mu)} \xrightarrow{P} Z,$$

4. Define $M_{\lambda,n} = \max(X_1, \ldots, X_n)$, where $X_1, \ldots, X_n$ are i.i.d. Exponential(λ) random variables and $\lambda > 0$.

   (a) Find the c.d.f. of $M_{\lambda,n}$.

   (b) Prove or disprove $M_{\lambda,n} \xrightarrow{P} \infty$ as $n \to \infty$ (λ is fixed).

   (c) Prove or disprove $M_{\log(n),n} \xrightarrow{P} 1$ in probability as $n \to \infty$.

(d) Find a sequence c_n such that $c_n(M_{\log(n),n} - 1) \xrightarrow{\mathcal{D}} S$, where S is non-degenerate. What is the c.d.f. of S?
(Hint: The following fact from calculus might be useful:

$$\left(1 - \frac{1}{n^{1+s/\log n}}\right)^n \to e^{-e^{-s}}.)$$

(e) Prove or disprove that $\lambda M_{\lambda,n} - c_n \xrightarrow{\mathcal{D}} S$ with the same c_n and S as in the previous part. (Hint: You can do this part even if you cannot do part 4d.)