STOR 634, CWE, 2010

1. (*15 points*) Show that if T is a measurable map from $(\Omega_1, \mathcal{F}_1)$ to $(\Omega_2, \mathcal{F}_2)$ and μ is a measure on $(\Omega_1, \mathcal{F}_1)$ then $\mu \circ T^{-1}$ is a measure on $(\Omega_2, \mathcal{F}_2)$.

2. (*15 points*) Let $\{f_n\}_{n\geq 1}$ and g be measurable functions on a probability space $(\Omega, \mathcal{F}, \mu)$, such that $|f_n| \leq g$ and g is integrable. Show that $\liminf_{n\to\infty} f_n$ and $\limsup_{n\to\infty} f_n$ are integrable and the following inequalities hold:

$$\int (\liminf f_n) d\mu \leq \liminf (\int f_n d\mu), \quad \int (\limsup f_n) d\mu \geq \limsup (\int f_n d\mu).$$

3. (*20 points*) Let $(\Omega, \mathcal{F}, \mu)$ be a probability space. Let $\{f_n\}, f$ be measurable functions such that $f_n \geq 0$ a.e. and $f \geq 0$ a.e. Also suppose that f_n and f are integrable. Define $\nu_n : \mathcal{F} \to \mathbb{R}_+$ and $\nu : \mathcal{F} \to \mathbb{R}_+$ as follows.

$$\nu_n(A) = \int_A f_n d\mu, \quad \nu(A) = \int_A f d\mu, \quad A \in \mathcal{F}$$

Note that ν_n and ν are finite measures. Suppose that $f_n \to f$ a.e. and $\nu_n(\Omega) \to \nu(\Omega)$. Prove, showing all steps, that

$$\sup_{A\in\mathcal{F}} |\nu(A) - \nu_n(A)| \to 0$$

as $n \to \infty$.

4. (*15 points*) Let $\{\mu_n\}$ be a sequence of probability measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that for each $\delta > 0$, $\mu_n(\{x : |x| \geq \delta\}) \to 0$ as $n \to \infty$. Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded Borel measurable function. Let

$$f_n(x) = \int f(x-y)\mu_n(dy), \quad x \in \mathbb{R}, n \geq 1.$$

Show that if f is uniformly continuous then $f_n \to f$, uniformly on $\mathbb{R}$.

5. (*35 points*)

a. (*15 pts*) For a signed measure ν, give definitions of $|\nu|$, ν^+ and ν^-.

b. (*20 pts*) Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Let $f \in L_1(\Omega, \mathcal{F}, \mu)$. Note that ν_f defined below is a signed measure.

$$\nu_f(A) = \int_A f d\mu, \quad A \in \mathcal{F}.$$

Show that, for $A \in \mathcal{F}$,

$$\nu_f^+(A) = \int_A f^+ d\mu, \quad \nu_f^-(A) = \int_A f^- d\mu,$$

and $|\nu_f|(A) = \int_A |f| d\mu$.