August 13, 2014 Name:________________________

COMPREHENSIVE WRITTEN EXAM – STOR654 MATHEMATICAL STATISTICS

All problem parts have equal weight. In budgeting your time expect that some part will take longer than others. When solving multi-part problems feel free to use results of earlier parts even if you cannot solve them in proving later parts. Do not forget to split the time between both papers.

1. Let $X_1, \ldots, X_n$ be i.i.d. Bernoulli(p). In what follows we will consider the loss function $L(p, a) = \frac{(p-a)^2}{p}$.

   (a) Find the Bayes rule for the loss function $L(p, a)$ and Beta(α, β) prior for p.

   (b) Find the MINIMAX estimator of p for this loss function.

2. Let $X_1, \ldots, X_n$ be i.i.d. Exponential(λ). We only observe $Y = \min\{X_1, \ldots, X_n\}$. We do not observe the Xs. In what follows, derive general answers and evaluate them numerically for $n = 10, Y = 0.02$.

   (a) What is the distribution of Y?

   (b) Find the 95% HPD credible interval for λ using an improper prior $\pi(\lambda) = \lambda^{-1}$.

   (c) Find the 90% equal tailed two sided confidence interval for λ. Explain how you found it.

   (d) Compute the Bayes factor for $\mathcal{H}_0 : \lambda \geq 20$ vs. $\mathcal{H}_1 : \lambda < 20$ using the proper prior $\lambda \sim \Gamma(1, 1)$. What is your decision?

   (e) Propose and evaluate a p-value for $\mathcal{H}_0 : \lambda \geq 20$ vs. $\mathcal{H}_1 : \lambda < 20$. What is your decision?

   (f) Modify the prior from part 2d to test $\mathcal{H}_0 : \lambda = 20$ vs. $\mathcal{H}_1 : p \neq 20$ and compute the Bayes factor. What is your decision?

   (g) Consider the likelihood ratio test for testing $\mathcal{H}_0 : \lambda = 20$ vs. $\mathcal{H}_1 : \lambda \neq 20$. Evaluate the value of the test statistic and find p-value of the likelihood ratio test. If you cannot find closed form of the solution, describe how you would proceed to obtain numerical solution.

3. Let $\begin{pmatrix} X_1 \\ Y_1 \end{pmatrix}, \ldots, \begin{pmatrix} X_n \\ Y_n \end{pmatrix}$ be i.i.d. $N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right)$, $\rho \in (-1, 1)$.

(a) Find a minimal sufficient statistics for this model. (Hint: The statistic is 2 dimensional.)

(b) Is this model an exponential family? If yes, find the natural 2 parameter exponential family.

(c) Prove that both $\sum_{i=1}^n X_i^2$ and $\sum_{i=1}^n Y_i^2$ are separately an ancillary statistics. Is the random vector $\left(\sum_{i=1}^n X_i^2, \sum_{i=1}^n Y_i^2\right)'$ jointly an ancillary statistics?