STOR 634 questions (2013/14 CWE)

Problem 1. Let X be a whole space. Suppose A_n, $n = 1, \dots, N$, is a finite partition of X, that is, A_n's are pairwise disjoint and $\sum_{n=1}^{N} A_n = X$. Show that a function $f : X \to \mathbb{R}$ is $\sigma(\{A_n, n = 1, \dots, N\})$-measurable if and only if f is constant on each A_n.

Problem 2. Let μ be a measure on a field $\mathcal{F}$. Show that the “distance” $d(A, B) = \mu(A \triangle B)$, $A, B \in \mathcal{F}$, satisfies the triangle inequality, that is, $d(A, B) \le d(A, C) + d(C, B)$ for $A, B, C \in \mathcal{F}$.

Problem 3. Suppose that f_n converges to f in measure on $(X, \mathcal{S}, \mu)$. Let $g : \mathbb{R} \to \mathbb{R}$ be a Borel function. (a) If g is continuous on $\mathbb{R}$ and $\mu(X) < \infty$, show that $g(f_n)$ converges to $g(f)$ in measure. (b) Show with an example that (a) is incorrect in general when $\mu(X) = \infty$.

Problem 4. Let $(X, \mathcal{S}, \mu)$ be a measure space, f an integrable function and $E_n = \{x \in X : |f(x)| \ge n\}$, $n \ge 1$. (a) Show that if E is the set where f is not finite, then

$$\mu(E) = \lim_{n \to \infty} \mu(E_n) = 0.$$

(b) Show also the following stronger property: $\lim_{n \to \infty} n\mu(E_n) = 0$.

Problem 5. Suppose F and G are right-continuous, non-decreasing functions on $[a, b]$, $-\infty < a < b < \infty$. (a) Show that

$$\int_{(a,b]} G(x)dF(x) = F(b)G(b) - F(a)G(a) - \int_{(a,b]} F(x-)dG(x).$$

(b) Give an example of F ,G, $[a, b]$ for which the formula does not hold if $F(x-)$ is replaced by $F(x)$ in the last integral.

Problem 6. Suppose $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{A}_3$ are independent classes of events, each closed under intersections and, without loss of generality, containing Ω. Let $\mathcal{B}_1 = \sigma(\mathcal{A}_1)$, $\mathcal{B}_2 = \sigma(\mathcal{A}_2)$ and $\mathcal{B}_3 = \sigma(\mathcal{A}_3)$. Show that $\mathcal{B}_1$, $\mathcal{B}_2$ and $\mathcal{B}_3$ are also independent classes of events.

All six problems carry equal weight. *Good luck!*