Statistics 654 Comprehensive Written Exam
August, 2012

1. Let U, V be random variables with finite second moment, and let $\mathrm{SD}(\cdot)$ denote the usual standard deviation.

a. Show that $\mathrm{SD}(U+V) \leq \mathrm{SD}(U) + \mathrm{SD}(V)$.

b. Show that $\mathbb{E}|U| + \mathbb{E}|V| \leq \sqrt{2}(\mathbb{E}U^2 + \mathbb{E}V^2)^{1/2}$

2. Let $\mathcal{P} = \{f_\theta(x) : \theta \in \Theta\}$ be a family of probability mass functions for a discrete random variable $X \in \mathcal{X}$.

a. Define what it means for a statistic $T : \mathcal{X} \to \mathbb{R}^d$ to be sufficient for θ.

b. Carefully state the Factorization Theorem for sufficient statistics in the discrete setting above.

c. The Factorization Theorem is of an "if and only if" form. Prove one direction of the theorem.

3. Let X denote an i.i.d. sample $X_1, \ldots, X_n$ from the $\mathcal{N}(\mu, \sigma^2)$ distribution.

a. What are the maximum likelihood estimates of μ and σ based on X? You may write down the estimates without deriving them. Are these estimates unbiased?

b. Establish the numerical identity $\sum_{i=1}^n (x_i - \theta)^2 = \sum_{i=1}^n (x_i - \overline{x})^2 + n(\overline{x} - \theta)^2$, valid for real numbers $x_1, \ldots, x_n, \theta$.

Suppose now that we wish to test the hypothesis $\mathrm{H}_0 : \mu \leq \mu_0$ vs. $\mathrm{H}_1 : \mu > \mu_0$ for a fixed number μ_0, and that the population variance σ^2 of our sample is unknown.

c. Carefully identify the natural t-statistic $T(x)$ for this test. Does one reject H_0 for large or small values of T?

d. Show as carefully as you can that testing based on $T(x)$ is equivalent to a likelihood ratio test

e. Define the p-value associated with an observed statistic $T(x)$, and show that it has a simple, closed form.

Suppose now that we wish to test the hypothesis $H_0 : \mu = \mu_0$ vs. $H_1 : \mu > \mu_0$, again assuming that the variance σ^2 is unknown.

f. Let $\alpha \in (0, 1)$. Identify a one-sided level-α test for H_0 vs. H_1. Be sure to establish that the level of the test is indeed α.

g. Use the test in the part f to find a $1 - \alpha$ confidence interval for μ. Carefully identify the steps in your argument.