# STOR 634 questions (2011/12 CWE)

*Problem 1.* By definition, $\mathcal{B}^2 = \mathcal{B} \times \mathcal{B} = \sigma\{B_1 \times B_2 : B_1, B_2 \in \mathcal{B}\}$. Show that $\mathcal{B}^2 = \sigma\{(a_1, b_1] \times (a_2, b_2] : a_1 \leq b_1, a_2 \leq b_2\}$. (As usual, $\mathcal{B} = \mathcal{B}(\mathbb{R})$ consists of the Borel sets on $\mathbb{R}$.)

*Problem 2.* Let $[a]$ be the integer part of $a \in \mathbb{R}$, that is, the largest integer which is smaller than or equal to $a$. Let also $\{a\} = a - [a]$ be the fractional part of $a \in \mathbb{R}$. Show that the functions $f(x) = [2x]$ and $g(x) = \{2x\}$, $x \in \mathbb{R}$, are Borel measurable.

*Problem 3.* Let $(X, \mathcal{S}, \mu)$ be a finite measure space. Suppose that $f_n \to f$ in measure. If $\mu(f_n = 0) = \mu(f = 0) = 0$, show that $1/(f_n)^2 \to 1/(f)^2$ in measure. Provide a direct argument, and not an indirect argument based on (sub)subsequences.

*Problem 4.* Let $(X, \mathcal{S}, \mu)$ be a measure space, and $f \in L^1(X, \mathcal{S}, \mu)$. Show that

$$\sup_{A:\mu(A)<\delta} \left| \int_A f d\mu \right| \to 0, \quad \text{as } \delta \to 0.$$

*Problem 5.* Let $(X, \mathcal{S}, \mu)$ be a $\sigma$-finite measure space and $f : X \to \mathbb{R}$ a non-negative measurable function. Let also $G$ be a non-decreasing, right-continuous function on $[0, \infty)$ with $G(0) = 0$, and $\mu_G$ be the Lebesgue-Stieltjes measure on $\mathcal{B}[0, \infty)$ associated with $G$. Show that

$$\int_X G(f(x))\mu(dx) = \int_{[0,\infty)} \mu(f \geq s)\mu_G(ds).$$

*Problem 6.* Let $\{\xi_i\}_{i\geq 1}$ be a sequence of identically distributed random variables and let $M_n = \max\{|\xi_j| : 1 \leq j \leq n\}$. If $E|\xi_1|^\alpha < \infty$ for some $\alpha \in (0, \infty)$, then show that

$$\frac{M_n}{n^{1/\alpha}} \to 0 \quad \text{a.s.}$$

All six problems carry an equal weight. *Good luck!*