Statistics 654 Comprehensive Written Exam
August, 2011

1. Let X, Y, Z be non-negative random variables with finite third moments. Consider the following inequality:

$$\mathbb{E}(XYZ) \leq (\mathbb{E}X^3)^{1/3}(\mathbb{E}Y^3)^{1/3}(\mathbb{E}Z^3)^{1/3} \tag{1}$$

a. Show that (1) holds if X, Y, Z are independent. Make sure to justify each step of your argument.

b. Show that (1) holds in general, even if X, Y, Z are not independent. Make sure to justify each step of your argument.

2. Let X be a real-valued random variable distributed according to one of two densities, f_0 or f_1, each of which is positive on the real line. Consider the problem of testing $\mathrm{H}_0 : X \sim f_0$ vs. $\mathrm{H}_1 : X \sim f_1$ with zero-one loss.

a. Specify the elements of the decision theoretic model for the testing problem described above.

b. Let $T(x) = f_1(x)/f_0(x)$. Show that T is sufficient for the testing problem.

c. What can you say about the optimality properties of the test $d(x) = I(T(x) \geq 1)$?

d. Show that the sum of the Type I and Type II errors of the test $d(x)$ defined in part (c) is equal to $1 - \frac{1}{2}\mathrm{E}_0|1 - f_1(X)/f_0(X)|$.

3a. Find the Fisher information for a Poisson(λ) distribution.

b. Show that $\{\text{Poisson}(\lambda) : \lambda > 0\}$ is an exponential family.

c. Let $X_1, \ldots, X_n$ be i.i.d. with $X_i \sim \text{Poisson}(\lambda)$. Provide two essentially different arguments to show that the sample mean $\overline{X}$ is a best unbiased estimate of λ.

4. Let $0 < \theta < \infty$ and let X be a random variable with density $f(x|\theta) = \theta\, x^{\theta-1}$ for $0 \leq x \leq 1$ and $f(x|\theta) = 0$ otherwise.

a. Find the density of $Y = -\log X$.

Suppose that $X_1, \ldots, X_n$ are i.i.d. with $X_i \sim f(x|\theta)$.

b. Find the method of moments estimator of θ.

c. Find the maximum likelihood estimate $\hat{\theta}_{MLE}$ of θ.

d. What can you say about the distribution of $\hat{\theta}_{MLE}$?

e. Determine whether or not the estimate $\hat{\theta}_{MLE}$ is biased.

August, 2011 Name:_______________________

COMPREHENSIVE WRITTEN EXAM – STOR655 MATHEMATICAL STATISTICS

All problem parts have equal weight. In budgeting your time expect that some part will take longer than others. When solving multi-part problems feel free to use results of earlier parts even if you cannot solve them in proving later parts. Do not forget to split the time between both STOR654 and STOR655 papers.

1. Let $X_1, \ldots, X_n$ be i.i.d. $U(0, \theta)$, $\theta > 0$.

   (a) Show that $U = \theta^{-1} X_{(n)}$ is a pivot; recall $X_{(n)} = \max\{X_1, \ldots, X_n\}$. What is the density of U?

   (b) Find the 95% confidence interval for θ based on this pivot. When presented by choices, select the shortest possible interval.

2. Let $X_1, X_2, \ldots, X_n, \ldots$ be i.i.d. with the following probability mass function:

$$p(x; p, \lambda) = (1-p) I_{\{0\}}(x) + p \frac{e^{-\lambda} \lambda^{x-1}}{(x-1)!} I_{\{1,2,\ldots\}}(x),$$

   where $\Theta = \{\theta = (p, \lambda) : 0 \le p \le 1, \lambda > 0\}$. Denote $\mathbf{X}_n = (X_1, \ldots, X_n)$ and $\bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i$. Recall

$$\mathcal{L}(\theta; \mathrm{x}_n) = \prod_{i=1}^n \left[(1-p) I_{\{0\}}(x_i) + p \frac{e^{-\lambda} \lambda^{x_i - 1}}{(x_i - 1)!} (1 - I_{\{0\}}(x_i)) \right] I_{\{0,1,2,\ldots\}}(x_i).$$

   (a) Find the MLE of (p, λ).

   (b) Prove that

$$\sqrt{n} \left(\frac{\bar{X}_n}{1 - \frac{\sum_{i=1}^n I_0(X_i)}{n}} - 1 - \lambda \right) \xrightarrow{d} Y,$$

   where $Y \sim \mathrm{N}(0, \lambda/p)$. (Hint: Use the bivariate CLT based on $\begin{pmatrix} X_1 \\ I_0(X_1) \end{pmatrix}, \begin{pmatrix} X_2 \\ I_0(X_2) \end{pmatrix}, \ldots$ and the delta method)

(c) Find the GLR, Λ_n for testing $\mathcal{H}_0 : p = 1/2$ versus $\mathcal{H}_1 : p \neq 1/2$. What is the asymptotic distribution of $-2 \log \Lambda_n$ under $\mathcal{H}_0$?

(d) Find the GLR, Λ_n^* for testing $\mathcal{H}_0 : p = 0$ versus $\mathcal{H}_1 : p > 0$. What is the distribution of $-2 \log \Lambda_n^*$ under $\mathcal{H}_0$? (Hint: Think before blindly applying a theorem. The regularity conditions are not met in this case!)

(e) Assume that the value of $p = p_0$ is known. Give an asymptotic distribution of the MLE of λ.

(f) Consider a prior on (p, λ) such that $p \sim \text{Beta}(1, 1)$, $\lambda \sim \Gamma(1, 1)$, p and λ are independent. Find the posterior. What is the posterior Bayes estimator of p and λ?

(g) Modify the prior and find the Bayes factor for testing $\mathcal{H}_0 : p = 1/2$ versus $\mathcal{H}_1 : p \neq 1/2$.