COMPREHENSIVE WRITTEN EXAMINATION, PAPER III
FRIDAY AUGUST 20, 2010, 9:00 A.M. – 1:00 P.M.
STOR 664 Question

Suppose we have an experiment to fit the Michaelis-Menten relationship

$$y_i = \frac{\beta_1 x_i}{\beta_2 + x_i} + \epsilon_i, \quad i = 1, 2, ..., n,$$

where $\epsilon_i \sim N[0, \sigma^2]$ (independent for each i). Suppose the ordinary least squares estimates are $\hat{\beta}_1, \hat{\beta}_2$, that s^2 is the estimate of σ^2, and assume $\mathrm{Cov}\begin{pmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{pmatrix} \approx \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix} \sigma^2$.

(a) (30 points) Show that, as an approximation, we may take

$$a_{11} = \sum \frac{x_i^2}{(\hat{\beta}_2 + x_i)^4} \Big/ \left[\sum \frac{x_i^2}{(\hat{\beta}_2 + x_i)^2} \sum \frac{x_i^2}{(\hat{\beta}_2 + x_i)^4} - \left\{ \sum \frac{x_i^2}{(\hat{\beta}_2 + x_i)^3} \right\} \right]$$

and give corresponding expressions for a_{12}, a_{22}.

For the remainder of the question, assume that the traditional linear model distribution theory for $(\hat{\beta}_1, \hat{\beta}_2, s^2)$ is also valid in this case — in particular, if k_1 and k_2 are constants and $\mathrm{Var}(k_1\hat{\beta}_1 + k_2\hat{\beta}_2) = k_3^2\sigma^2$ for some constant k_3, then $(k_1\hat{\beta}_1 + k_2\hat{\beta}_2)/k_3 s$ has a t_{n-2} distribution. Suppose we are interested in determining the value $x = x^*$ for which the reaction rate $\beta_1 x^*/(\beta_2 + x^*)$ is some predetermined value c.

(b) (30 points) Suggest an estimator $\hat{x}^*$, and calculate its approximate standard error by the delta method.

(c) (40 points) Show how it is possible to construct a $100(1-\alpha)\%$ confidence set for x^* by means of a sequence of tests of the hypothesis $H_0 : x^* = x_0$ (for given x_0) at significance level α. Assume t^* is the $(1 - \alpha/2)$-quantile of the t_{n-2} distribution. Derive a condition (in terms of $\hat{\beta}_1$, $\hat{\beta}_2$, c, etc.) for this confidence set to be a confidence interval, and show that in that case, the length of the interval is

$$\frac{2c\sqrt{\left\{\hat{\beta}_2(\hat{\beta}_1 - c) - s^2 t^{*2} a_{12}\right\}^2 - \left\{(\hat{\beta}_1 - c)^2 - s^2 t^{*2} a_{11}\right\}(\hat{\beta}_2^2 - s^2 t^{*2} a_{22})}}{(\hat{\beta}_1 - c)^2 - s^2 t^{*2} a_{11}}$$

What happens in the cases where the confidence set is not an interval?