August, 2010 Name:____________________

STOR 654 EXAM

The eighteen parts have equal weight. In budgeting your time expect that some parts will take longer than others. When solving multi-part problems feel free to use results of earlier parts even if you cannot solve them in proving later parts.

1. A ball is colored in such a way that 10% of its surface is blue and the rest is white. Show that regardless of the paint distribution you can always inscribe a cube in such a way that all its vertices are colored white. (Hint: Consider what would happen if the cube was picked at random.)

2. Let X and Y be any random variables with finite variances. Also, it is assumed that $E[X|Y] = Y$, $E[Y|X] = X$. Prove or disprove that, $P(X = Y) = 1$.

3. Let X and Y be discrete random variables with joint mass function $f_{X,Y}(x, y)$.

   (a) Show that $E\log(f_X(X)) \geq E\log(f_Y(X))$.

   (b) Show that
   $$I = E\left(\log\left(\frac{f_{X,Y}(X,Y)}{f_X(X)f_Y(Y)}\right)\right)$$
   satisfies $I \geq 0$ with equality if and only if X and Y are independent.

4. Assume that $\begin{pmatrix} X \\ Y \end{pmatrix}, \begin{pmatrix} X_1 \\ Y_1 \end{pmatrix}, \begin{pmatrix} X_2 \\ Y_2 \end{pmatrix}, \ldots$ are i.i.d. random vectors, where both X and Y are positive random variables with finite moments of all orders. Denote the raw moments $\mu'_{r,s} = EX^rY^s$, $r = 0, 1, 2\ldots$, $s = 0, 1, 2, \ldots$.

   (a) Find the limiting distribution of $\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^n X_iY_i - \mu'_{1,1}\right)$.

   (b) Find the limiting distribution of $\frac{n\left(\frac{1}{n}\sum_{i=1}^n X_iY_i - \mu'_{1,1}\right)^2}{\left(\frac{1}{n}\sum_{j=1}^n X_j^2Y_j^2\right) - \left(\frac{1}{n}\sum_{k=1}^n X_kY_k\right)^2}$.

(c) Find the approximate distribution of $(\bar{X}_n)(\bar{Y}_n)$.

5. Let $U_1, U_2, \ldots$ be iid χ_1^2 random variables, and let $M \in \{1, 2, \ldots\}$ be independent of $\{U_i\}$ and have pmf $p(\cdot)$. Define $X = \sum_{j=1}^{M} U_j$.

   (a) Show that X has density $f(x) = \sum_{j=1}^{\infty} p(j) g_j(x)$, where g_j is the density of a central χ^2 random variable with j degree of freedom.

   (b) Show that if $N \sim \text{Poisson}(\frac{1}{2}\theta^2)$ is independent of $\{U_i\}$ then $X = \sum_{j=1}^{2N+1} U_j$ has density $f(x) = \sum_{j=0}^{\infty} P(N = j)\, g_{2j+1}(x)$. Conclude that $X \stackrel{\mathcal{L}}{=} Y^2$ where $Y \sim N(\theta, 1)$.

   (c) Find the density of a non-central $\chi^2(p, \lambda)$ distribution.

   (d) Show that if $Y \sim \chi^2(p, \lambda)$ then $EY = p + 2\lambda$, $\text{Var}(Y) = 2(p + 4\lambda)$.