

STOR 655 problems for CWE August 2009

(1) Let X_1, X_2, \dots be iid samples from a Pareto population having density

$$f(x|\lambda, \theta) = \theta \lambda^\theta x^{-(1+\theta)}, \quad x > \lambda$$

with parameters $\lambda > 0$ and $\theta > 0$. Write $X^n = (X_1, \dots, X_n)$ for a sample of size n and $x^n = (x_1, \dots, x_n)$ for a realization of X^n .

(1a) Let

$$U_n = \min_{1 \leq i \leq n} X_i \quad \text{and} \quad V_n = \prod_{i=1}^n X_i$$

. Show that (U_n, V_n) is a minimal sufficient statistic for (λ, θ) based on X^n .

(1b) Show that the MLE of (λ, θ) based on X^n is given by

$$\begin{aligned} \hat{\lambda} &= U_n \\ \hat{\theta} &= \left(\frac{1}{n} \log V_n - \log U_n \right)^{-1} \end{aligned}$$

(1c) Show that the MLE $(\hat{\lambda}, \hat{\theta})$ given in (1b) is consistent.

(1d) For $\alpha \in (0, 1)$ and large n , construct a confidence interval for λ with an approximate level $1 - \alpha$. Justify your answer.

(2) Let X_1, X_2, \dots be iid samples from a Bernoulli population with $P(X_1 = 1) = \theta = 1 - P(X_1 = 0)$. Assume a mixture prior π on $\theta \in [0, 1]$: with a point mass $P(\theta = 1/2) = p$ and a continuous component with probability $1 - p$, i.e. θ follows a uniform distribution over the interval $[0, 1]$, where $p \in (0, 1)$ is a known constant. Consider a Bayesian test (with 0-1 loss) for $H_0 : \theta = 1/2$ vs $H_1 : \theta \neq 1/2$.

(2a) Verify the posterior probability $\pi(\theta = 1/2 | x^n) = \frac{p/2^n}{m(x^n)}$ where the marginal probability

$$m(x^n) = p/2^n + (1 - p) \int_0^1 \theta^{s_n} (1 - \theta)^{n - s_n} d\theta$$

with $s_n = x_1 + \dots + x_n$.

(2b) For the special case $n = 2$, $p = 1/4$, determine whether H_0 is rejected when $s_2 = 1$ is observed.

(2c) In general, show that if the observation x^n satisfies

$$g(s_n/n) < \frac{p}{1 - p}$$

then H_0 is accepted, where $g(\theta) = 2^n \theta^{s_n} (1 - \theta)^{n - s_n}$.

Note that this is a sufficient condition, not necessary. Asymptotic argument for deriving an approximate critical region with large n is NOT a part of this test.