STOR 635, CWE 2009
Exam consists of nine questions. Total points: 100.
Name:

1. **(5 points)** Let $\{X_n\}_{n\geq 1}$ be an i.i.d. sequence such that $\mathbb{E}(X_1) = 0$ and $Var(X_1) = 1$. What can you say about $\mathbb{P}(\sum_{n=1}^{\infty} X_n$ converges)? Give detailed reasons.

2. **(10 points)** Suppose X, Y, Z, U are square integrable random variables on some probability space such that (X, U) and (Y, U) have the same probability law. Suppose that $Y = \mathbb{E}(Z \mid U)$ a.s. Show that $X = Y$ a.s.

3. **(15 points)** Let $\{X_n\}_{n\geq 0}$ be a sequence of nonnegative integrable random variables adapted to some filtration $\{\mathcal{F}_n\}$. Let $\{\tau_k\}_{k\geq 1}$ be a sequence of $\mathcal{F}_n$ stopping times such that $\tau_k \uparrow \infty$ as $k \to \infty$. Suppose that, for each $k \geq 1$, $\left\{Y_n^k = X_{n\wedge\tau_k}, n \geq 0\right\}$, is a martingale. Show that $\{X_n\}_{n\geq 0}$ is a supermartingale. If additionally, $\mathbb{E}(X_n) = \mathbb{E}(X_0)$ for each n, then show that $\{X_n\}_{n\geq 0}$ is a martingale.

4. **(10 points)** State the *Martingale Convergence Theorem*. Show by constructing an example that if the main assumption of the theorem is not satisfied then the conclusion of the theorem may fail.

5. **(10 points)** Give an example of a sequence $\{X_n\}_{n\geq 1}$ of random variables that is L^1 bounded but not uniformly integrable.

6. **(15 points)** Let $\{X_n\}_{n\geq 0}$ be a martingale with $X_0 = 0$ and let N be a stopping time. State a theorem precisely that gives general conditions under which $\mathbb{E}(X_N) = 0$. Let $\{\xi_n\}_{n\geq 1}$ be i.i.d., mean 0 random variables such that $|\xi_n| \leq C$, for each $n \geq 1$. Suppose $X_k = \xi_1 + \cdots + \xi_k$, for $k \geq 1$ and $X_0 = 0$. Let $N = \inf\{k : X_k > 1\}$. Can $\mathbb{E}(N)$ be finite? Give reasons.

7. **(10 points)** Give an example of a sequence of probability measures $\{\mu_n\}$ on some Polish space S such that $\mu_n \Rightarrow \mu$ but for some open subset G of S, $\liminf_{n\to\infty} \mu_n(G) > \mu(G)$.

8. **(15 points)** The following is an incomplete statement of Scheffe's Theorem: *Let f_n, f be nonnegative measurable functions on $\mathbb{R}$ such that $f_n(x) \to f(x)$, as $n \to \infty$, for each $x \in \mathbb{R}$. Suppose that $\int_{\mathbb{R}} f_n(x)\,dx = 1$ for each $n \geq 1$. Then $f_n \to f$ in L^1.* Show by constructing an example that the above statement is false in general. Give the complete and correct statement of Scheffe's Theorem.

9. **(10 points)** Show that an i.i.d. sequence $\{X_n\}_{n\geq 0}$ of $\mathbb{R}$ valued random variables is tight.