STOR 655 problems for CWE August 2008

Let $X_1, X_2, ...$ be iid samples from a 3-parameter Weibull population, denoted by W3, with density

$$f(x|\alpha,\beta,\theta) = (\beta/\alpha)\ [(x-\theta)/\alpha]^{\beta-1} \exp\{-[(x-\theta)/\alpha]^{\beta}\}, \quad x > \theta$$

with scale parameter $\alpha > 0$, shape parameter $\beta > 0$ and location parameter (or range parameter) $\theta \in \mathbb{R}$. Write $X^n = (X_1, ..., X_n)$ for a sample of size n and $x^n = (x_1, ..., x_n)$ for a realization of X^n.

(a) For the special case with known β and θ, show that the W3 density can be expressed as a canonical form $f^*(y|\eta)$ of an exponential family.

(b) Let $L(\alpha, \beta, \theta|x^n)$ denote the likelihood function of (α, β, θ) based on x^n in W3. Show that there exists a *profile likelihood* function of (β, θ) defined by

$$L^*(\beta, \theta|x^n) = \sup_{\alpha>0} L(\alpha, \beta, \theta|x^n).$$

(c) Consider the special case with $\theta = 0$, $\alpha = 1$ and $n \geq 2$. Does there exist a minimal sufficient statistic for β whose dimensionality is less than n? Explain why.

(d) For the special case with $\beta = 1$, find a minimal sufficient statistic for (α, θ) based on X^n.

(e) Assume $\beta \in (0, 1)$ is known. Find a MLE $(\hat{\alpha}, \hat{\theta})$ of (α, θ) based on X^n.

(f) For the special case with known β and θ [same as part(a)] and large n, find an approximate 95% confidence interval for α.

(g) Assume $\beta = 1$ and write $\lambda = 1/\alpha$. Let λ and θ have independent priors, as an exponential distribution with mean 1 and a uniform distribution over the interval $(0, 1)$ respectively. Under the squared error loss, find a Bayes estimator for (λ, θ) based on X^n. Hint: Recall the gamma integral $\Gamma(\nu) = \int_0^\infty x^{\nu-1}e^{-x}\ dx$, $\nu > 0$ satisfies $\Gamma(\nu + 1) = \nu\ \Gamma(\nu)$.