Statistics 654 Comprehensive Written Exam
August, 2008

1. State and prove the version of the weak law of large numbers presented in class.

2. Suppose that $Y_1, \ldots, Y_n$ are random variables such that $Ee^{sY_i} \leq e^{s^2/2}$ for each $s \geq 0$. Show that

$$\Theta \;=\; E\left[\max_{1\leq i\leq n} Y_i\right] \;\leq\; \sqrt{2\ln n}$$

Hint: Begin by considering $e^{s\Theta}$ for a value $s > 0$.

3. Let $X_1, X_2, \ldots, X$ be random variables, all defined on the same probability space, such that X_n converges to X in probability. Establish the following relations directly, without appealing to results from class.

a. Show that $X = O_P(1)$.

b. Show that $X_n = O_P(1)$.

c. Show that $O_P(1) \cdot o_P(1) = o_P(1)$.

4. Let $X_1, X_2, \ldots, X$ be identically distributed non-negative random variables such that EX^2 is finite. Note that the X_i may be dependent.

a. Show that $EXI\{X > \alpha\} \to 0$ as $\alpha \to \infty$.

b. Show carefully that for $\alpha > 0$,

$$\max_{1\leq i\leq n} X_i \cdot I\left\{\max_{1\leq i\leq n} X_i \geq \alpha\right\} \;\leq\; \max_{1\leq i\leq n} X_i \cdot I\{X_i \geq \alpha\}$$

c. Show that

$$n^{-1}E\left[\max_{1\leq i\leq n} X_i\right] \to 0 \;\text{ as }\; n \to \infty.$$

5. For $n \geq 1$ let X_n have a Bern(n,p) distribution, where $p \in (0,1)$ is fixed. What can you say about the limiting distribution of the random vectors $Y_n = (X_n, n - X_n)^T$ after suitable scaling and/or shifting?

6 . Let $X_1, \ldots, X_n$ be an i.i.d. sample from a population with $EX = \mu$ and $\mathrm{Var}(X) = \sigma^2 < \infty$. Let $\bar{X}_n = n^{-1} \sum_{i=1}^{n} X_i$ be the sample mean of $X_1, \ldots, X_n$.

a. After suitable scaling and/or shifting, what can you say about the limiting distribution of $(\bar{X}_n)^3$ when $\mu \neq 0$?

b. After suitable scaling and/or shifting, what can you say about the limiting distribution of $(\bar{X}_n)^3$ when $\mu = 0$?

7. Let X, Y be random variables such that EX^2 and EY^2 are finite. Let $SD(\cdot)$ denote standard deviation. Show that $\mathrm{SD}(X+Y) \leq \mathrm{SD}(X) + \mathrm{SD}(Y)$.

8. If $X \sim \mathcal{N}_p(\mu, \Sigma)$, find the distribution of $AX + b$ where A is a $d \times p$ matrix and $b \in \mathbb{R}^d$.