Stor 635 questions (2007/08 CWE)

Problem 1. Let

$$\xi_n = \begin{cases} p_n, & \text{with prob. } 1 - \frac{1}{n}, \\ -p_n, & \text{with prob. } \frac{1}{n}, \end{cases} \qquad n \geq 1,$$

be independent random variables, where $p_n > 0$ and $p_n \to 0$, as $n \to \infty$. Find necessary and sufficient conditions for convergence of the series $\sum_{n=1}^{\infty} \xi_n$ in terms of the sequence $\{p_n\}_{n\geq 1}$.

Problem 2. Let ϕ be a characteristic function of a random variable ξ. (a) If $|\phi(t_0)| = 1$ for some $t_0 > 0$, show that, for some $a \in \mathbb{R}$, $\xi \in \{a + 2\pi n/t_0,\ n \in \mathbb{Z}\}$ a.s. (b) If there are $t_k \to 0$ ($t_k > 0$) such that $|\phi(t_k)| = 1$, show that $\xi = \text{const}$ a.s.

Problem 3. Let

$$\xi_n = \begin{cases} \pm n, & \text{with prob. } \frac{p_n}{2}, \\ 0, & \text{with prob. } 1 - p_n, \end{cases} \qquad n \geq 1,\ p_n > 0,$$

be independent random variables. Give an example of $\{p_n\}_{n\geq 1}$ for which $\{\xi_n\}_{n\geq 1}$ satisfies the Lindeberg condition. State the resulting convergence according to Central Limit Theorem.

Problem 4. This problem concerns basic properties of conditional expectations given σ-field. (a) Use the Radon-Nikodym Theorem to show existence of $E(\xi|\mathcal{G})$. (b) What is $E(\xi|\mathcal{G})$ for $\mathcal{G} = \sigma\{E_0, E_1, E_2, \ldots\}$ with mutually disjoint events E_k such that $P(E_0) = 0$, $P(E_k) > 0$, $k = 1, 2, \ldots$, and $\Omega = \cup_{k=0}^{\infty} E_k$. ($c$) What is $E(g(\xi)|\eta)$ if a vector (ξ, η) has a density $f(x, y)$ such that $f(x, y) > 0$ for all $x, y \in \mathbb{R}$ (as usual, it is supposed that $E|g(\xi)| < \infty$).

Problem 5. Let $\{\xi_i^n, i, n \geq 1\}$ be i.i.d. nonnegative, integer-valued random variables. Consider a branching process $\{Z_n\}_{n\geq 1}$ defined by $Z_0 = 1$ and

$$Z_{n+1} = \begin{cases} 0 & \text{if } Z_n = 0, \\ \xi_1^{n+1} + \ldots + \xi_{Z_n}^{n+1} & \text{if } Z_n > 0. \end{cases}$$

Let $\mathcal{F}_n = \sigma\{\xi_i^m, i \geq 1, m \leq n\}$ and suppose $\mu = E\xi_i^m \in (0, \infty)$. Show that ($a$) $\{Z_n/\mu^n, \mathcal{F}_n\}$ is a martingale, and that (b) it converges a.s.