

# COMPREHENSIVE WRITTEN EXAMINATION, PAPER II

## STOR 655 problems for CWE August 2007

Let  $X_i$ ,  $i = 1, 2, \dots$  be a sequence of iid random variables with the common density

$$f(x|\theta) = \theta\gamma x^{\gamma-1} e^{-\theta x^\gamma}, \quad x \geq 0,$$

where  $\theta > 0$  is an unknown parameter and  $\gamma > 0$  is a known constant. Write  $X^n = \{X_1, \dots, X_n\}$ .

- (1) Find the MLE  $T_n$  of  $g(\theta) = 1/\theta$  based on  $X^n$ .
- (2) Calculate  $E_\theta T_n$  and  $Var_\theta T_n$ .
- (3) Derive the Cramér-Rao lower bound for unbiased estimators of  $g(\theta)$ .
- (4) Find the UMVUE for  $g(\theta)$ .
- (5) For  $0 < \alpha < 1$  and  $\theta_0 > 0$ , derive the uniformly most powerful test of size  $\alpha$  for  $H_0 : \theta \leq \theta_0$  vs  $H_1 : \theta > \theta_0$ .
- (6) For large  $n$ , find a large sample approximation of the critical region for the test in (5).
- (7) Let  $Y_i$ ,  $i = 1, 2, \dots$  be iid samples from the conditional density of  $X_1$  given  $X_1 > b$  where  $b > 0$  is known. Write  $Y^n = \{Y_1, \dots, Y_n\}$ . Find the MLE of  $g(\theta) = 1/\theta$  based on  $Y^n$ .
- (8) Assume that  $\theta$  follows a prior exponential distribution with mean 1. Under the squared error loss, find a Bayes estimator for  $\theta$  based on  $Y^n$ . Hint: Recall the gamma function  $\Gamma(\beta) = \int_0^\infty x^{\beta-1} e^{-x} dx$ ,  $\beta > 0$  satisfies  $\Gamma(\beta + 1) = \beta \Gamma(\beta)$ , and in particular,  $\Gamma(n) = (n - 1)!$ .