Statistics 164 Comprehensive Written Exam
August, 2007

1. Let $X, Y \geq 0$ be random variables such that $XY \geq 1$. Show that $EX \cdot EY \geq 1$.

2. Let $Z = \{z_{i,j} : 1 \leq i, j \leq n\}$ be an $n \times n$ binary matrix whose entries $z_{i,j}$ are i.i.d. Bernoulli(1/2) random variables, so that $P(z_{i,j} = 1) = 1/2 = P(z_{i,j} = 0)$. For every pair of sets $A, B \subseteq \{1, \ldots, n\}$ with $|A| = |B| = k$, there is an associated $k \times k$ submatrix of Z defined by

$$C(A : B) = \{z_{i,j} : i \in A, j \in B\}.$$

Here A corresponds to a set of k rows, B corresponds to a set of k columns, and $C(A, B)$ contains the entries of Z that belong both to a row in A and a column in B. Call $C(A, B)$ a submatrix of 1s if all of its entries are equal to 1.

a. For $k = 1, \ldots, n$, let $U_{k,n}$ be equal to the number of $k \times k$ submatrices of 1s in Z. What are the possible values of $U_{k,n}$ for fixed k? How does $U_{k,n}$ change as k increases? Carefully show that $E(U_{k,n}) = \binom{n}{k}^2 2^{-k^2}$.

b. Using the result of part (a), find a simple bound on $P(U_{k,n} \geq l)$ for integers $l \geq 1$.

c. Let $M(Z)$ be the largest k such that Z contains a $k \times k$ submatrix of 1s. Show that $P(M(Z) \geq k) \leq n^{-2r}$ when $k = 2\log_2 n + r$. [Hint: in your calculations, the simple bound $\binom{n}{k} \leq n^k$ may help.]

3. Let X, Y be independent random variables with $Y > 0$. Find simple bounds on $E(X/Y)$ and $\ln(1/Ee^{XY})$ involving the expectations of X and Y. You may assume all relevant expectations exist.

4. The version of Slutsky's theorem presented in class has two parts, one concerning a sum of random variables, and the other concerning a product. Carefully state both parts of Slutsky's theorem, and prove the part concerning a sum of random variables.

5. Let $X_1, \ldots, X_n$ be an i.i.d. sample from a population with $EX = \mu$ and $\text{Var}(X) = \sigma^2 < \infty$. Let $\bar{X}_n = n^{-1} \sum_{i=1}^n X_i$ be the sample mean of $X_1, \ldots, X_n$, and define $T_n = \bar{X}_n^2$.

a. After suitable scaling and/or shifting, what can you say about the asymptotic distribution of T_n when $\mu \neq 0$?

b. After suitable scaling and/or shifting, what can you say about the asymptotic distribution of T_n when $\mu = 0$?

6. Let $a, b > 0$. For each $n \geq 1$, let $\mathbf{X}_n = (X_{1,n}, \ldots, X_{n,n})$ be a $\mathcal{N}_n(\mathbf{0}, \Sigma_n)$ random variable, where Σ_n has diagonal entries equal to $a+b$ and off-diagonal entries equal to b. Find the limiting distribution of $\bar{X}_n = n^{-1} \sum_{i=1}^n X_{i,n}$ as n tends to infinity.

7. Suppose that $U_n \to U$ in probability and $X_n \Rightarrow X$ in law. Assume each random variable is defined on the same probability space. In each of the following cases, indicate the type of convergence as n tends to infinity (if any) and the limit. In each case, briefly justify your answer.

a. $\sqrt{1 + X_n^2}$

b. $V_n + X_n^3$

c. $X_n \sqrt{1 + n^{-1} U_n}$

d. $U_n / (1 + |U_n|)$