CWE 2006-07, STOR 635, Instructor: Budhiraja

Total points: **100**.

1. (**15 points.**) Precisely state and prove the Monotone Convergence Theorem for Conditional Expectations.

2. (**20 points.**) Let $\{X_n, \mathcal{F}_n\}$ be a uniformly integrable sub-martingale on some probability space. Show that for any stopping time N, the family $\{X_{n\wedge N}, n \geq 1\}$ is uniformly integrable.

3.(**15 points.**) Let $\{\mu_n, n \geq 1\}$, μ be probability measures on $(\mathbb{R}, \mathbb{B}(\mathbb{R}))$ such that μ_n converge weakly to μ. Show that the sequence $\{\mu_n, n \geq 1\}$ is tight.

4. (**10 + 15 + 5 points.**) Let $\{X_n\}$ be an independent sequence of mean 0, variance 1 random variables. Let $S_n = X_1 + \cdots X_n$ and suppose that $\frac{S_n}{\sqrt{n}}$ converges in distribution to a standard normal random variable.

(a) Show that for every $x \in \mathbb{R}$,

$$\limsup_{n\to\infty} \mathbb{P}(S_n > \sqrt{n}x) \leq \mathbb{P}(S_n > \sqrt{n}x,\ i.o.) \leq \mathbb{P}(S \geq x),$$

where $S = \limsup_{n\to\infty} \frac{S_n}{\sqrt{n}}$.

(b) By using Kolmogorov's $0-1$ law show that $\limsup_{n\to\infty} \frac{S_n}{\sqrt{n}} = +\infty$ a.s. [Hint: Recall that if a r.v. U is measurable with respect to the tail sigma-field of an independent sequence of random variables then for some $c \in [-\infty, \infty]$, $\mathbb{P}(U = c) = 1$.]

(c) Recall from Skorohod's representation theorem, one can find random variables $\tilde{S}_n$ (having same distribution as S_n) and a standard normal random variable Z given on a common probability space such that $\lim_{n\to\infty} \frac{\tilde{S}_n}{\sqrt{n}} = Z$ a.s. Why does'nt this contradict the statement in part (b) of the problem?

5. (**20 points.**) Let $X_n \geq 0$ be independent for $n \geq 1$. Show that the following are equivalent.
(i) $\sum_{n=1}^{\infty} X_n < \infty$ a.s.
(ii) $\sum_{n=1}^{\infty} (\mathbb{P}(X_n > 1) + \mathbb{E}(X_n 1_{X_n \leq 1})) < \infty$
(iii) $\sum_{n=1}^{\infty} \mathbb{E}(\frac{X_n}{1+X_n}) < \infty$.