Stat 165 problems for CWE August 2006

Let (X_i, Y_i), $i = 1, 2, ...$ be a sequence of iid 2D random vectors with the following common distribution: X_1 follows a uniform distribution over the interval $(-\theta, \theta)$, and $Y_1 = X_1^k$, where θ is a positive real number and k is a positive integer. For a positive integer n, we write $(X, Y)^n = \{(X_1, Y_1), ..., (X_n, Y_n)\}$, $X^n = \{X_1, \ldots, X_n\}$, and $Y^n = \{Y_1, ..., Y_n\}$.

(1) Let θ be an unknown parameter and $k = 2$. Show that X_1 and Y_1 are dependent.

(2) Following the assumptions in (1), does it make sense to estimate the correlation ρ between X_1 and Y_1 based on a joint sample $(X, Y)^n$? If "yes", then propose an estimator for ρ; if "no", explain why.

(3) Assume both θ and k are unknown parameters. Derive the general formula for EX_1^m for an arbitrary positive integer m, based on which construct estimators for θ and k based on a sample Y^n using method-of-moments. Make additional assumptions if needed.

(4) Show that the density for Y_1 is given by

$$f_{Y_1}(y) = \begin{cases} \frac{1}{2\theta k}\, y^{\frac{1}{k}-1}, & -\theta^k < y < 0 \text{ or } 0 < y < \theta^k, \quad \text{if } k \text{ is odd;} \\ \frac{1}{\theta k}\, y^{\frac{1}{k}-1}, & 0 < y < \theta^k, \quad \text{if } k \text{ is even.} \end{cases}$$

(5) Show that (U_n, V_n) is a 2D minimal sufficient statistic for (θ, k) based on a sample Y^n, where

$$U_n = \max_{1 \le i \le n} |Y_i| \quad \text{and} \quad V_n = \prod_{i=1}^{n} |Y_i|.$$

(6) Define $\alpha = \theta^k$ and $\beta = 1/k$. Find a MLE $(\hat{\alpha}, \hat{\beta})$ for (α, β) based on Y^n.

(7) Assume $k = 2$ and θ follows a prior uniform distribution over the interval $[1, 2]$. Find the Bayes estimator for θ based on Y^n under the absolute error loss.

(8) Assume $k = 2$ and θ follows a prior uniform distribution over the interval $[1, 2]$. Conduct the Bayesian test based on Y^n for the hypotheses $H_0 : \theta \le 3/2$ vs $H_1 : \theta > 3/2$.

(9) Show the following consistency result: U_n given in (5) converges to α in probability with an exponential rate as $n \to \infty$.