Statistics 164 Comprehensive Written Exam
August, 2006

1. Recall that the Euclidean norm of a vector $a = (a_1, \ldots, a_m) \in \mathbb{R}^m$ is given by $||a|| = \sqrt{a_1^2 + \cdots + a_m^2}$. The Cauchy-Schwartz inequality states that for $a, b \in \mathbb{R}^m$,

$$a^T b \;=\; a_1 b_1 + \cdots + a_m b_m \;\leq\; ||a||\,||b||.$$

a. Prove the Cauchy-Schwartz inequality starting, from the basic fact that

$$0 \leq \left(\frac{a_j}{||a||} - \frac{b_j}{||b||} \right)^2$$

for each $j = 1, \ldots, m$.

b. Let $\mathbf{X} = (X_1, \ldots, X_m)$ be an m-dimensional random vector with $E\mathbf{X} = \mathbf{0}$ and with variance-covariance matrix Σ. Suppose that for some number $\kappa > 0$, the matrix $\Sigma - \kappa I$ is non-negative definite. Show that for each vector $a \in \mathbb{R}^m$

$$||a||^2 \;\leq\; \frac{1}{\kappa} a^T \Sigma a$$

Hint: Write the left hand side of the inequality as a quadratic form.

c. Show that the right hand side of the inequality in part (b) is at most $\kappa^{-1}\,||a||^2\,E||\mathbf{X}||^2$.

2. Let Y be a random variable taking values in the two point set $\{-1, 1\}$ and let $p = P(Y = 1)$.

a. Find the value of c that minimizes Ee^{-cY}.

b. Suppose now that (X, Y) are jointly distributed with X taking values in a finite set $\mathcal{X}$ and $Y \in \{-1, +1\}$. Let $p(x) = P(Y = 1|X = x)$. Using part (a), find the function f^* that minimizes $Ee^{-Yf(X)}$ over all functions $f : \mathcal{X} \to \mathbb{R}$. Justify your answer.

3. Let $Z \sim \mathcal{N}(0, 1)$ and define $Y = |Z|$.

a. Find the moment generating function of Y in as simple a form as possible.

b. Find the density of Y.

4. Let Poiss(λ) denote a Poisson distribution with parameter $\lambda \geq 0$, and let $X_n \sim$ Poiss(λ_n).

a. Show that $X_n \to 0$ in probability if $\lambda_n \to 0$.

b. Find the characteristic function of a Poission(λ) random variable.

c. Show that $X_n \Rightarrow$ Poiss(λ) if $\lambda_n \to \lambda$. Justify your answer.

5. Suppose that $U_n \to U$ in probability, $V_n \to V \geq 0$ in probability and $X_n \Rightarrow X$ in law. Assume each random variable is defined on the same probability space. In each of the following cases, indicate the type of convergence as n tends to infinity (if any) and the limit. In each case, briefly justify your answer.

a. $\sqrt{1 + |X_n|}$

b. $n^{-1/2}U_n + X_n$

c. $V_n^2 + X_n^2$

d. $(2 + n^{-1/4}U_n)\, X_n$

e. $U_n/(1 + V_n)$

6. Let $X_1, X_2, \ldots$ be i.i.d. with $EX_i = \mu > 0$ and $\text{Var}(X_i) = \sigma^2$. What can you say about the limiting distribution of $Y_n = (n^{-1} \sum_{i=1}^{n} X_i)^{1/3}$ after suitable normalization?