Stat 154 questions (2005/06 CWE)

Problem 1. Let μ be a finite measure on $(X, \mathcal{S})$ where $\mathcal{S} = \sigma(\mathcal{E})$ with a field $\mathcal{E}$. Show that, for $A \in \mathcal{S}$ and any $\epsilon > 0$, there is $A_\epsilon \in \mathcal{E}$ such that $|\mu(A) - \mu(A_\epsilon)| \leq \mu(A \triangle A_\epsilon) < \epsilon$. Show, in particular, that the first inequality in the latter relation holds.

Problem 2. Without providing any proofs, describe the steps in the construction of the Lebesgue-Stieltjes measure on the real line (including the completion step). If the Lebesgue-Stieltjes measure μ is a point mass at 0, what is $\mathcal{B}_\mu$, the σ-field obtained by completing the σ-field $\mathcal{B}$ of Borel sets under μ?

Problem 3. (a) Provide an example of a nonmeasurable function f on some measurable space $(X, \mathcal{S})$ such that f^2 is measurable. (b) If f and g are measurable functions on a measurable space $(X, \mathcal{S})$, show that $f + g$ is also measurable.

Problem 4. Let $X = \mathbb{R}$, $\mathcal{S} = \mathcal{B}(\mathbb{R})$, $\mu =$ Lebesgue measure. Consider the function

$$f(x) = \begin{cases} 0, & x < 0 \\ 1/2^k, & 2k \leq x < 2k+1 \\ -1/3^k, & 2k+1 \leq x < 2k+2, \quad k = 0, 1, 2, \ldots. \end{cases}$$

Compute $\int_X f(x)\mu(dx)$ by using the definition of integral. Give an example of another measure μ on $(\mathbb{R}, \mathcal{B})$ for which this integral is not defined.

Problem 5. Let μ be the Lebesgue measure on $(\mathbb{R}, \mathcal{B})$ and μ_F be the Lebesgue-Stieltjes measure on $(\mathbb{R}, \mathcal{B})$ induced by the function $F(x) = x^{2005}$, $x \in \mathbb{R}$. Show that $\mu_F \ll \mu$ and determine the Radon-Nikodym derivative $d\mu_F/d\mu$. Is is true that $\mu_F \sim \mu$?

Problem 6. Give an example of a sequence of measurable functions f_n on some measure space $(X, \mathcal{S}, \mu)$ such that (at the same time)

- f_n does not converge μ-a.e. on X,
- f_n converges in measure μ, and
- f_n converges in $L^p(X, \mathcal{S}, \mu)$ for $p < 2006$ but not for $p > 2006$.

Good luck!