STATISTICS 165 WRITTEN QUESTION

August 2005 Gordon Simons

A certain type of astronomical observation X, based on theoretical considerations, should have an exponential distribution with an unknown mean $\theta > 0$. The astronomer's objective is to estimate θ as accurately as possible. Now suppose that $X_1, X_2, \ldots$ are independent and identically distributed *potential* astronomical observations, distributed as X, but, due to instrumental limitations, a potential observation X_i is not even observable if it is smaller than an unknown threshold value $\theta_1 > 0$. Thus, in truth, there is only a subsequence $Y_1, Y_2, \ldots$ of the X_i's available to the astronomer for observation. Of these, the astronomer observes $Y_1, \ldots, Y_n$ and, ignoring the instrumental limitations, attempts to estimate θ with $\bar{Y}_n = (Y_1 + \cdots + Y_n)/n$.

(a) It is intuitively obvious that the observations $Y_1, \ldots, Y_n$ in the astronomer's sample are independent and identically distributed. Don't try to show this, here, but do show that the common pdf of the Y_i's is

$$f(y|\theta, \theta_1) = \theta^{-1}\, e^{-(y-\theta_1)/\theta} \text{ for } y \geq \theta_1 \text{ } (= 0 \text{ elsewhere}),$$

and calculate the bias in the astronomer's estimator.

(b) Find the method of moments estimator of (θ, θ_1), and call it $(\hat{\theta}, \hat{\theta}_1)$.

(c) Compare the expectation of $\hat{\theta}^2$ with θ^2.

(d) Find a minimal sufficient statistic for (θ, θ_1).

(e) Find the maximum likelihood estimator of (θ, θ_1), and call it $(\tilde{\theta}, \tilde{\theta}_1)$.

(f) Compute the bias of the estimator $\tilde{\theta}$ if it is used to estimate θ.

(g) Show that $\hat{\theta}$ and $\tilde{\theta}$ are consistent estimators of θ.

While you are not being asked to evaluate the variances of the estimators of θ, the quality of these estimators, as measured by their variances, most certainly do depend on the value of θ_1. Interestingly, it is possible to estimate θ consistently *even when* θ_1 *exceeds* θ.