Stat 155: Comprehensive Exam 2005

Note: The only questions that (perhaps) should take more than 10 minutes are questions 4 and 6. Budget your time wisely.

1. Let $\{X_n\}$ be a sequence of random variables.
(a) Say what is meant by the tail σ field of the sequence $\{X_n\}$. State Kolmogorov's $0-1$ law.

(b) Let $S_n = X_1 + \cdots X_n$. Let $0 < c_n \to \infty$ be a sequence of reals.
Show that $\{\limsup_{n\to\infty} S_n/c_n > x\}$ is in the tail σ field of $\{X_n\}$ for any real number x.

2. (a) State the Borel-Cantelli lemmas.
(b) Let $\{X_i\}$ be i.i.d. with $\mathbb{E}(|X_1|) < \infty$. Let $0 < c_n \to \infty$ be a sequence of reals. Define $Y_n = X_n 1_{|X_n| \le n}$, $S_n = X_1 + \cdots X_n$ and $T_n = Y_1 + \cdots Y_n$. Show that S_n/c_n converges almost surely iff T_n/c_n does.

3. (a) State precisely the definition of a Martingale and a stopping time.
(b) Show that if $\{\mathcal{F}_n\}$ is some filtration and $A \in \mathcal{F}_k$ for some k then $\tau \doteq k1_{A^c} + (k+1)1_A$ is a $\{\mathcal{F}_n\}$ stopping time.
(c) Let $\{X_n\}$ be a sequence of integrable $\{\mathcal{F}_n\}$ adapted random variables. Show that it is a martingale iff there exists some $c \in \mathbb{R}$ such that $\mathbb{E}(X_\tau) = c$ for all bounded $\{\mathcal{F}_n\}$ stopping times τ. You may use the result (without proof) that for a martingale $\{X_n\}$ the sequence $\{X_{n\wedge\tau}\}$ is a martingale as well.

4. If $X_n \to X$ in probability then show that the following are equivalent.
(a) $\{X_n\}$ is u.i.
(b) $X_n \to X$ in L^1 as $n \to \infty$.
(c) $\mathbb{E}|X_n| \to \mathbb{E}|X|$ as $n \to \infty$.

5. (a) State the Martingale Convergence Theorem.
(b) Use the result in 5. to show that for a $\{\mathcal{F}_n\}$-martingale the following are equivalent.
(i) It is u.i.
(ii) It converges a.s. and in L^1.
(iii) It converges in L^1.
(iv) There is an integrable r.v. X such that $X_n = E(X \mid \mathcal{F}_n)$ for all n.

6. (a) State precisely the notion of tightness and weak convergence of probability measures.
(b) Recall the key "Continuity Theorem": If $\{\mu_n\}$ is a sequence of probability measures on $\mathbb{R}$ with ch.f. $\{\phi_n\}$ then: (i) If μ_n converge weakly to some μ then $\phi_n(t)$ converges to $\phi(t)$ for all t where ϕ is the ch.f. associated with μ; (ii) If ϕ_n converges to some ϕ pointwise and ϕ is continuous at 0 then μ_n converges weakly to some probability measure μ.

Prove (i) and say clearly, in as much detail you can, what are the key steps in the proof of (ii).